

INTERIOR POINT METHODS: Current Status and Future Directions

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Introduction and Synopsis

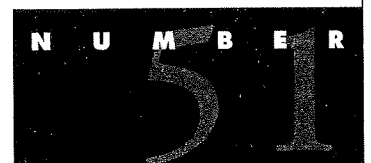
The purpose of this article is twofold: to provide a synopsis of the major developments in interior point methods for mathematical programming in the last twelve years for the researcher who is unfamiliar with interior points, and to discuss current and future research directions in interior point methods for researchers who have some familiarity with interior point methods. Throughout the article, we provide brief selective guides to the most lucid relevant research literature as a means for the uninitiated to become acquainted (but not overwhelmed) with the major developments in interior point methods.

Interior point methods in mathematical programming have been the largest and most dramatic area of research in optimization since the development of the simplex method for linear programming. Over the last twelve years, interior point methods have attracted some of the very best researchers in operations research, applied mathematics, and computer science. Approximately 2,000 papers have been written on the subject following the seminal work of Karmarkar [10]. See for example the *netlib* electronic library initiated by Eberhard Kranich, and the World Wide Web interior point archive:

<http://www.mcs.anl.gov/home/otc/InteriorPoint/archive.html> .

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This article provides a synopsis of the major developments in interior point methods for mathematical programming in the last twelve years, and discusses current and future research directions in interior point methods, with a brief selective guide to the research literature.



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$$E_x = \{x \in \mathbb{R}^n \mid Ax = b, (x - \bar{x})^T \bar{x}^{-2} (x - \bar{x}) \leq 1\}$$

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Interior point methods have permanently changed the landscape of mathematical programming theory, practice, and computation. Linear programming is no longer synonymous with the celebrated simplex method, and many researchers now tend to view linear programming more as a special case of nonlinear programming due to these developments.

The pedagogy of interior point methods has lagged the research on interior point methods until quite recently, partly because these methods (i) use more advanced mathematical tools than do pivoting/simplex methods, (ii) their mathematical analysis is typically much more complicated, and (iii) the methods are less amenable to geometric intuition. For most of the last twelve years, educators have struggled with issues of how and where to introduce interior point methods into the curriculum of linear and/or nonlinear programming, and how to cogently exposit interior point methods to students (and fellow researchers). As the research on interior point methods for linear programming has settled down (and the research on interior points for nonlinear programming has heated up), a number of new book projects on linear programming and/or interior point methods have recently appeared on the scene which promise to surmount these pedagogical difficulties. For example, in the last two years alone, the following new book projects on linear programming have been undertaken which contain substantive and rigorous treatments of interior point methods: *Linear Programming: A Modern Integrated Analysis* by Romesh Saigal (Kluwer, 1995), *Interior Point Methods in Mathematical Programming* edited by Tamas Terlaky (Kluwer, 1996), *Introduction to Linear Optimization* by Dimitris Bertsimas and John Tsitsiklis (Athena Scientific, forthcoming), *Linear Programming: Foundations and Extensions* by Robert Vanderbei (Kluwer, forthcoming), *Interior Point Algorithms: Theory and Analysis* by Yinyu Ye (John Wiley, forthcoming), *Primal-Dual Interior Point Algorithms* by Stephen Wright (SIAM, forthcoming), and *Linear Programming I: An Introduction* by George Dantzig and Mukund Thapa (Springer Verlag, forthcoming).

To begin our synopsis of interior point methods for linear programming, we consider the linear programming problem in standard form:

$$\begin{array}{ll} P: & \text{minimize} & c^T x \\ & \text{s.t.} & Ax = b \\ & & x \geq 0, \end{array}$$

where x is a vector of n variables, whose standard linear programming dual problem is:

$$\begin{array}{ll} D: & \text{maximize} & b^T y \\ & \text{s.t.} & A^T y + s = c \\ & & s \geq 0. \end{array}$$

Given a feasible solution x of P and a feasible solution (y, s) of D , the duality gap is simply $c^T x - b^T y = x^T s \geq 0$.

We introduce the following notation which will be very convenient for manipulating equations, etc. A feasible solution x of P is *strictly feasible* if $x > 0$, and a feasible solution (y, s) of D is *strictly feasible* if $s > 0$. Let e denote the vector of ones, i.e., $e = (1, \dots, 1)^T$. Suppose that $x > 0$. Define the matrix X to be the $n \times n$ diagonal matrix whose diagonal entries are precisely the components of x .

Then $Xe = x$, and $X^{-1}e = (1/x_1, \dots, 1/x_n)^T$. Also, notice that both X and X^{-1} are positive-definite symmetric matrices.

There are many different types of interior point algorithms for linear programming, with certain common mathematical themes having to do with the logarithmic barrier function. In the authors' opinions, most interior point algorithms fall into one of three main categories: affine scaling methods, potential reduction methods, and central trajectory methods. We now briefly summarize these three categories.

Affine Scaling Methods. The basic strategy of the affine scaling algorithm is as follows: given a strictly feasible solution \bar{x} of P , construct a simple local ellipsoidal approximation of the feasible region of P that is centered at \bar{x} . Call this ellipsoid $E_{\bar{x}}$. Then, optimize the objective function $c^T x$ over $E_{\bar{x}}$, and use the resulting direction with a suitable steplength to define a new algorithmic iterate. The specifics of this strategy are as follows. Given a strictly feasible solution \bar{x} of P , the *Dikin Ellipsoid* at \bar{x} is defined as:

$$E_{\bar{x}} = \{x \in \mathcal{R}^n \mid Ax = b, (x - \bar{x})^T \bar{X}^{-2} (x - \bar{x}) \leq 1\}.$$

(It is straightforward to show that $E_{\bar{x}}$ is always contained in the feasible region of P whenever \bar{x} is strictly feasible.) The affine scaling direction at \bar{x} is then the solution to the following direction-finding problem:

$$\begin{array}{ll} (ADFP_{\bar{x}}): & \text{minimize} & c^T d \\ & \text{s.t.} & Ad = 0 \\ & & d^T \bar{X}^{-2} d \leq 1. \end{array}$$

Note that $(ADFP_{\bar{x}})$ is a convex program with all linear components except for one convex quadratic constraint. It can be solved directly by forming the associated Karush-Kuhn-Tucker system and then solving an associated linear equation system. One can also write down a closed form solution algebraically after a little bit of matrix manipulation. Letting \bar{d} denote the solution to problem $(ADFP_{\bar{x}})$ the next iterate of the affine scaling algorithm is obtained by setting $x_{\text{new}} = \bar{x} + \alpha \bar{d}$, where the steplength α is chosen by one of several strategies to ensure the strict feasibility of the new iterate x_{new} while maintaining a suitable improvement in the objective function.

The affine scaling algorithm is attractive due to its simplicity and its good performance in practice. (However, its performance is quite sensitive to the starting point of the algorithm.) The proof of convergence of the algorithm in the absence of degeneracy is fairly straightforward, but under degeneracy, such a proof is surprisingly long and difficult. There have not been any results on bounds on the efficiency of the algorithm, but it is suspected (for very good reasons that are beyond the scope of this synopsis) that the algorithm is exponential time in the worst case.

Some variants/extensions on the basic affine scaling algorithm are the dual affine scaling algorithm (designed to work on the dual problem D), as well as various versions that work simultaneously on the primal and on the dual, using a variety of ellipsoidal constructions in the space of the primal and the dual. Two comprehensive references on affine scaling methods are Tsuchiya [19] and the book by Saigal [17].

Potential Reduction Methods. Potential reduction methods typically are designed to find improving solutions to the following optimization problem:

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$$\begin{aligned} \text{PRP: minimize} \quad & f(x,y,s) = q \ln(c^T x - b^T y) - \sum_{j=1}^n \ln(x_j) \\ \text{s.t.} \quad & Ax = b \\ & x > 0, \\ & A^T y + s = c \\ & s \geq 0, \end{aligned}$$

where the objective function $f(x,y,s)$ is called the *potential function*, and q is a parameter of the potential function. It was this type of problem that Karmarkar introduced in his seminal paper [10]. Notice that the "first part" of the potential function is q times the logarithm of the duality gap, and we would like to drive this part to $-\infty$. The second part of the potential function is the logarithmic barrier function, which is designed to repel feasible solutions from the boundary of the feasible region. The potential function is a surrogate for the goal of reducing the duality gap to zero, and under some mild assumptions regarding the linear program P , one can easily show that the duality gap is bounded from above by a function of the value of the potential function, i.e.,

$$c^T x - b^T y \leq C_1 e^{f(x,y,s)/q}$$

for a constant C_1 that is problem specific.

Now, suppose that the parameter q has been set. In a typical potential reduction method, we have a current iterate $(\bar{x}, \bar{y}, \bar{s})$ and we seek new iterate values $(x_{\text{new}}, y_{\text{new}}, s_{\text{new}})$ with a suitable decrease in the potential function. There are a number of tools that can be used to accomplish this, such as Newton's method, a "partial" Newton's method that only accounts for the Hessian of the second part of the potential function, and projective transformation methods combined with projected steepest descent. In a typical potential reduction algorithm, there is a guaranteed decrease in the potential function $f(x,y,s)$ of at least an amount δ at each iteration, where $\delta > 0$. Then, from the above, the duality gap is therefore decreased by a fixed proportion in at most q/δ iterations. This reasoning is then used to establish an upper bound on the total number of iterations needed to obtain a near-optimal solution within some optimality tolerance ϵ from some starting point (x^0, y^0, s^0) , namely

$$\frac{q}{\delta} \left(\ln \left(\frac{c^T x^0 - b^T y^0}{\epsilon} \right) + C_2 \right)$$

iterations, for a constant C_2 that depends on the problem P and on the starting point (x^0, y^0, s^0) . This type of logic underlies most potential reduction algorithms.

Although potential reduction methods do not have the simplicity of affine scaling methods, they are more attractive than affine scaling algorithms for at least two reasons: they have a performance guarantee, and they always produce dual information and so allow the user to specify an optimality tolerance to the computer. Also, with a line-search of the potential function, they can be made very efficient in practice.

(Karmarkar's original algorithm [10] used a very specific form of PRP and used the machinery of projective transformations in the algorithm and in the proofs of the algorithm's performance guarantees. Despite their original mystique, projective transformations are not necessary for potential reduction algorithms to work either in theory or in practice. However, in the authors' opinions, the framework of projective transformations is nevertheless of para-

mount importance, at least conceptually, in the understanding of interior point methods in general.)

There are numerous types of potential reduction methods, some using the potential function above, others using the so-called Tanabe-Todd-Ye symmetric primal-and-dual potential function

$$g(x,y,s) = q \ln(c^T x - b^T y) - \sum_{j=1}^n \ln(x_j) - \sum_{j=1}^n \ln(s_j),$$

which has additional desirable properties that go beyond this brief synopsis. In general the potential reduction methods all aim to drive a potential function to $-\infty$ by a variety of primal, dual, or primal-and-dual algorithmic tools. Almost all potential reduction methods enjoy good to excellent performance guarantees, i.e., complexity bounds. Potential reduction methods have not received much attention in terms of computational testing, due perhaps to early difficulties (which have since been overcome) of applying potential reduction methods in a combined Phase I-Phase II environment. For a comprehensive survey of potential reduction methods, see Anstreicher [4] or Todd [18].

Central Trajectory Methods. Interior point methods based on the *central trajectory* are by far the most useful in theory, the most used in practice, and, in our judgement, have the most aesthetic qualities. (In fact, one leading researcher has even referred to the central trajectory as "the most important object in modern optimization theory.") The *central trajectory* of the linear program P is obtained as the solution to an amended version of P , where a parameterized logarithmic barrier term is added to the objective function. Consider the logarithmic barrier problem $BP(\mu)$ parameterized by the positive barrier parameter μ :

$$\begin{aligned} BP(\mu): \quad \text{minimize} \quad & c^T x - \mu \sum_{j=1}^n \ln(x_j) \\ \text{s.t.} \quad & Ax = b \\ & x > 0. \end{aligned}$$

The Karush-Kuhn-Tucker conditions for $BP(\mu)$ are:

$$\begin{aligned} Ax = b, x > 0 \\ c - \mu X^{-1} e = A^T y. \end{aligned} \tag{1}$$

If we define $s = \mu X^{-1} e$, then we can rewrite these optimality conditions as:

$$\begin{aligned} Ax = b, x > 0 \\ A^T y + s = c, s > 0 \\ XSe - \mu e = 0. \end{aligned} \tag{2}$$

Let $(x(\mu), y(\mu), s(\mu))$ denote the solution to system (2) for the given positive parameter μ . Then the set $\Gamma = \{(x(\mu), y(\mu), s(\mu)) \mid \mu > 0\}$ is defined to be the *central trajectory* (also known as the central path) of the linear program P . From the first two equation systems of (2), we see that a solution (x, y, s) along the central trajectory is strictly feasible for the primal and the dual problem, and that the duality gap on the central trajectory is $x^T s = e^T XSe = \mu e^T e = \mu n$, which follows from the third equation system of (2). Substituting this equation in the third system of (2), we obtain the following equivalent and parameter-free characterization of the central trajectory:

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$$\begin{aligned} Ax &= b, x > 0 \\ A^T y + s &= c, s > 0 \\ XSe - (x^T s / n)e &= 0. \end{aligned} \quad (3)$$

The third equation system in (2) or (3) is precisely where the nonlinearity arises, and in general it is not possible to solve (2) or (3) in closed form except in trivial cases.

The strategy in most central trajectory methods is to solve for approximate solutions along the central trajectory (2) or (3) for a decreasing sequence of the duality gap (or equivalently, of the barrier parameter μ) that tends to zero in the limit. There are a number of ways to carry out this strategy. For example, for a given value of the duality gap or of the barrier parameter μ , one can choose to approximately optimize $BP(\mu)$ or, equivalently, to approximately solve (1), (2), or (3), or to approximately solve some other equivalent characterization of the central trajectory. Also, one can choose a number of ways to approximately solve the system of nonlinear equations under consideration (Newton's method is one obvious choice, as are predictor-corrector methods and other higher-order methods, preconditioned conjugate gradient methods, etc.). Overlayed with all of this is the way in which the numerical linear algebra is implemented. Furthermore, one needs to decide how to measure "approximate" in the approximate solution. Last of all, there is considerable leeway in developing a strategy for reducing the duality gap (or the barrier parameter μ) at each iteration. (For example, aggressively shrinking the duality gap seems like a good idea, but will also increase the number of iterations of Newton's method (or other method) that is used to resolve (approximately) the new system of nonlinear equations.)

In terms of theoretical performance guarantees, the best central trajectory methods are guaranteed to reduce the duality gap of the iterates by a fixed proportion in $O(\sqrt{n})$ iterations.

A short summary of central trajectory methods is given in Jansen et al. [9]. More comprehensive treatments of central trajectory methods are given in den Hertog [8] and the forthcoming book by Wright [22].

The rest of this article is devoted to a discussion of important current research topics in interior point methods. We discuss the following topics, in order: infeasible interior point methods, computational aspects of interior point methods, semidefinite programming, convex programming and self-concordance, homogeneous and self-dual methods, linear and nonlinear complementarity problems, and theoretical issues related to interior-point methods.

Infeasible Interior Point Methods

By definition, interior point methods naturally work on the interior (or relative interior) of a problem's feasible region, and consequently one obvious issue is how an initial feasible interior point can be obtained. Over the years, a number of techniques for handling the "feasibility" or "Phase I" problem have been proposed, including combined Phase I-Phase II methods, shifted-barrier methods, and homogeneous self-dual methods. In practice, methods based on a variation of a relatively simple algorithm, the "primal-dual infeasible-interior-point method" proved to be very successful. The basic method attempts to reduce the feasibility and

optimality gaps at each iteration by applying Newton's method to the system (2) or (3) from an initial point (x^0, y^0, s^0) which is not necessarily feasible for either P or D , i.e., $Ax^0 \neq b$ and/or $A^T y^0 + s^0 \neq c$, but which is "interior" in that $x^0 > 0$ and $s^0 > 0$. In this sense, the algorithm is a simple variant of the standard central trajectory path-following algorithm, but where the iterates are not necessarily feasible at each iteration. Let $(\bar{x}, \bar{y}, \bar{s})$ be an iterate. Then the Newton direction (d_x, d_y, d_s) for the algorithm is derived from the nonlinear equation system (2) and is the solution of the system

$$\begin{aligned} Ad_x &= -(A\bar{x} - b) \\ A^T d_y + d_s &= -(A^T \bar{y} + \bar{s} - c) \\ \bar{S}d_x + \bar{X}d_s &= -(\bar{X}\bar{S}e - \mu e). \end{aligned} \quad (4)$$

Of course, if the iterate $(\bar{x}, \bar{y}, \bar{s})$ is feasible, which is the case in a standard central trajectory interior point algorithm, then the right-hand-side of the first and second equations of (4) are 0, and consequently the directions d_x and d_s are orthogonal. As it turns out, the orthogonality of d_x and d_s is essential for an "easy" analysis of Newton's method for solving (2), and is lost when the iterate $(\bar{x}, \bar{y}, \bar{s})$ is infeasible.

Although quite successful computationally, the primal-dual infeasible-interior-point method long defied any reasonable theoretical analysis, nor even a proof of convergence, until a few years ago, when satisfactory analyses by several researchers emerged. One of the difficulties in the analysis of the algorithm was the lack of foreknowledge of the existence of the feasible solutions (interior or not), the existence of the central trajectory, or the existence of optimal LP solutions. In the case when the either P or D is infeasible, it is theoretically possible to detect the infeasibility of the problems P and/or D , but such detection mechanisms do not necessarily work well in practice.

To overcome these difficulties, another type of infeasible-interior-point algorithm has been developed. The basic algorithm uses the following variant of the system (3):

$$\begin{aligned} Ax &= b + \theta(Ax^0 - b), x > 0 \\ A^T y + s &= c + \theta(A^T y^0 + s^0 - c), s > 0 \\ XSe - \theta((x^0)^T s^0 / n)e &= 0 \end{aligned} \quad (5)$$

where (x^0, y^0, s^0) is the initiating point of the algorithm (where $x^0 > 0$ and $s^0 > 0$, but quite possibly $Ax^0 \neq b$ and $A^T y^0 + s^0 \neq c$), and $\theta \in (0, 1]$. Here, the goal is to use Newton's method to solve (5) for a decreasing sequence of values of θ tending to 0.

If $\|X^0 S^0 e - ((x^0)^T s^0 / n)e\|$ is small enough, the point (x^0, y^0, s^0) is a good approximate solution of the system (5) for $\theta=1$. The set of solutions to the system (5) forms a path parameterized by θ which does not lie in the feasible region of the problem unless the initial point is feasible. Nevertheless, if P and D are feasible, the path leads to optimal primal and dual solutions as θ goes to 0. If either P and/or D is infeasible, there exists a positive lower bound on θ for which the system (5) has a solution, and the path diverges to infinity as θ approaches this lower bound. By exploiting these and other features of the path, one can develop an infeasible interior point path-following algorithm which either solves P and D or detects infeasibility in a polynomial number of iterations.

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The former type of algorithms based on the Newton system (4) are preferred in practice, probably because they are more effective than the latter method (based on (5)) when the linear program is feasible. The authors believe, however, that the latter type of algorithm is most likely to outperform the former when the underlying linear program is either infeasible or is close to being infeasible. Affine scaling methods and potential reduction methods starting from an infeasible starting point have also been developed, but practical versions of these algorithms have not received very much attention. For a comprehensive summary of infeasible interior point algorithms, see Mizuno [14].

Computational Aspects of Interior Point Methods for Linear Programming

Much of the initial excitement about interior point methods stemmed from the rather remarkable computational promise of the method, as articulated by Karmarkar and others. Twelve years later, computational aspects of interior point methods are still of paramount importance. Although neither author is particularly qualified to comment on computational issues, it is only fair to briefly discuss key aspects of computation nevertheless.

After much effort in designing and testing various interior point methods for linear programming in the 1980s and early 1990s, the computational picture of interior point methods for linear programming has somewhat settled down. While a suitably customized simplex method implementation enjoys superior performance over interior point methods for most routine applications of linear programming, interior point algorithms are superior to the simplex method for certain important classes of problems. The simplex method tends to perform poorly on large massively-degenerate problems, whereas interior point methods are immune to degeneracy (and are aided by it in certain ways), and so one can expect an interior point algorithm to outperform the simplex method to the extent that the underlying linear program has massive degeneracy. This is the case in large scheduling problems, for instance, which give rise to LP relaxations of binary integer programs. Also, because the linear-algebra engine of an interior point iteration works with a Cholesky factorization of the matrix $\bar{A}\bar{X}^2\bar{A}^T$, interior point algorithms will outperform the simplex method to the extent that the matrix A is conducive to producing relatively sparse Cholesky factors of the matrix $\bar{A}\bar{X}^2\bar{A}^T$. Such is the case in large staircase multi-period linear programs, for example. Other than these two general problem types, there are not many other ways to predict which method will be more efficient. The "state-of-the-art" of interior point computation as of the early 1990s was described in the article of Lustig, Marsten, and Shanno [13]; more recent advances (using higher-order methods that are up to 25 percent faster) are described in Andersen et al. [3]. For a comprehensive treatment of computational issues in interior point methods, we recommend the forthcoming book by S. Wright [22].

There are a number of software codes for interior point methods for linear programming, including PCx (by Czyzyk, Mehrotra, and Wright), HOPDM (by Gondzio et al.), BPMPD (by Mészáros), OSL (IBM), CPLEX/Barrier (CPLEX Optimization Inc.), XPRESS-MP (Dash Associates), LOQO (by R. Vanderbei), and LIPSOL (by Y. Zhang). Information on these and other interior point method

codes is updated regularly on the World Wide Web page:

<http://www.mcs.anl.gov/home/wright/IPPD/>.

Some of these codes are free to the research community, others are solely commercial, and others are a hybrid.

Computational issues for nonlinear optimization are deferred to the following sections on semidefinite programming and on convex optimization.

Semidefinite Programming

In the opinion of the authors, semidefinite programming (SDP) is the most exciting development in mathematical programming in the 1990s. SDP has applications in traditional convex constrained optimization, as well as in such diverse domains as control theory and combinatorial optimization. Because SDP is solvable via interior point methods, there is the promise that these applications can be solved efficiently in practice as well as in theory. Before defining a semidefinite program, we need to amend our notation. Let S^n denote the set of symmetric $n \times n$ matrices, and let S_+^n denote the set of positive semidefinite (psd) $n \times n$ matrices. Then S_+^n is a closed convex cone in \mathcal{R}^n of dimension $n \times (n-1)/2$. We write " $X \geq 0$ " to denote that X is symmetric and positive semidefinite, and we write " $X \geq Y$ " to denote that $X - Y \geq 0$ (" \geq " is the Löwner partial ordering on S^n). Here, X is any symmetric matrix, not necessarily a diagonal matrix as denoted earlier. We write " $X > 0$ " to denote that X is symmetric and positive definite, etc. Let $X \in S^n$. A linear function $M(X)$ can be written as $M \bullet X$, where $M \bullet X = \sum_{i=1}^n \sum_{j=1}^n M_{ij} X_{ij}$. Then a semidefinite program (SDP) is an optimization problem of the form:

$$\begin{aligned} \text{SDP: minimize} \quad & C \bullet X \\ \text{s.t.} \quad & A_i \bullet X = b_i, \quad i=1, \dots, m \\ & X \geq 0, \end{aligned}$$

where X is an $n \times n$ matrix, and the data for the problem are the m symmetric matrices A_1, \dots, A_m , the symmetric matrix C , and the m -vector b . Notice that SDP has a linear objective function and linear equality constraints, just like a linear program. However, the standard LP constraint that x is nonnegative is replaced by the constraint that the variable X is symmetric and positive semidefinite. (It is helpful to think of $X \geq 0$ as stating that the *vector of eigenvalues* of X is nonnegative.) The Lagrange dual of SDP is derived as:

$$\begin{aligned} \text{SDD: maximize} \quad & b^T y \\ \text{s.t.} \quad & \sum_{i=1}^m y_i A_i + S = C \\ & S \geq 0. \end{aligned}$$

Given a feasible solution X of SDP and a feasible solution (y, S) of SDD, the duality gap is simply $C \bullet X - b^T y = X \bullet S \geq 0$.

As stated above, SDP has very wide applications in convex optimization. The types of constraints that can be modeled in the SDP framework include: linear inequalities, convex quadratic inequalities, lower bounds on matrix norms, lower bounds on determinants of symmetric positive semidefinite matrices, lower bounds on the geometric mean of a nonnegative vector, plus many others. Using these and other constructions, the following problems (among many others) can be cast in the form of a semidefinite pro-

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gram: linear programming, optimizing a convex quadratic form subject to convex quadratic inequality constraints, minimizing the volume of an ellipsoid that covers a given set of points and ellipsoids, maximizing the volume of an ellipsoid that is contained in a given polytope, plus a variety of maximum eigenvalue and minimum eigenvalue problems.

SDP also has wide applicability in combinatorial optimization as well. A number of *NP*-hard combinatorial optimization problems have convex relaxations that are semidefinite programs. In many instances, the *SDP* relaxation is very tight in practice, and in certain instances in particular, the *SDP* relaxation is provably quite tight and can be converted to provably very good feasible solutions with provably good bounds on optimal objective values. Last of all, *SDP* has wide applications in control theory, where a variety of control and system problems can be cast and solved as instances of *SDP*.

As it turns out, virtually *all* of the mathematics, constructions, and even the notation of interior point methods for linear programming extends directly to *SDP*. This is truly remarkable. (The extension of interior point methods to *SDP* was developed independently by Alizadeh [1] and Nesterov and Nemirovskii [15] using different frameworks.) For example, the analogous parameterized logarithmic barrier problem *BP*(μ) for linear programming extends to *SDP* as:

$$\begin{aligned} \text{BSDP}(\mu): \text{ minimize} \quad & C \bullet X - \mu \ln(\det(X)) \\ \text{s.t.} \quad & A_i \bullet X = b_i, \quad i=1, \dots, m, \\ & X > 0, \end{aligned}$$

where notice that $\ln(\det(X))$ replaces the logarithmic barrier function $\sum_{j=1}^n \ln(x_j)$. The optimality conditions for this problem can be written as:

$$\begin{aligned} A_i \bullet X &= b_i, \quad i = 1, \dots, m, \quad X > 0 \\ \sum_{i=1}^m y_i A_i + S &= C, \quad S > 0 \\ XS - \mu I &= 0, \end{aligned} \quad (6)$$

which should be compared with (2). The third equation system of (6) can alternatively be represented in many different equivalent ways, including for example, $(XS+SX)/2 - \mu I = 0$, resulting in many different non-equivalent Newton directions for solving (6).

In terms of theoretical performance guarantees, the best central trajectory methods for *SDP* are guaranteed to reduce the duality gap of the iterates by a fixed proportion in $O(\sqrt{n})$ iterations (where the variable X is an $n \times n$ matrix). This is identical to the theoretical performance guarantee for linear programming, even though the dimension of the variables in *SDP* is much larger ($n \times (n-1)/2$ as opposed to n for linear programming).

There are many very active research areas in semidefinite programming. In the arena of theory, there is research on the geometry and the boundary structure of *SDP* feasible regions (including notions of degeneracy). There is research related to the computational complexity of *SDP*, such as decidability questions, certificates of infeasibility, and duality theory. There is active research on *SDP* relaxations of combinatorial optimization in theory and in

practice. As regards interior point methods, there are a host of research issues, mostly involving the development of different interior point algorithms and their properties, including rates of convergence, performance guarantees, etc.

Because *SDP* has so many applications, and because interior point methods show so much promise, perhaps the most exciting area of research on *SDP* has to do with computation and implementation of interior point algorithms. Researchers are quite optimistic that interior point methods for *SDP* will become practical, efficient, and competitive. However, in the research to date, computational issues have arisen that are much more complex than those for linear programming, see for example Alizadeh, Haerberly, and Overton [2]. These computational issues are only beginning to be well-understood. They probably stem from a variety of factors, including the fact that the nonlinear conditions in the third equation system of (6) can be represented in very many different ways, resulting in many different Newton directions, see above. Other contributing issues might be the fact that *SDP* is not guaranteed to have strictly complementary optimal solutions (as is the case in linear programming), and the fact that the Jacobian of the *KKT* system defining the Newton step can be much more poorly conditioned than is the typical case in linear programming. Furthermore, there are challenges in developing the linear algebra (both symbolic and numeric) for solving *SDP* by interior point algorithms. Finally, because *SDP* is such a new field, there is no representative suite of practical problems on which to test algorithms, i.e., there is no equivalent version of the *netlib* suite of industrial linear programming problems.

A comprehensive survey of semidefinite programming is the article by Vandenberghe and Boyd [20].

Convex Programming and Self-Concordance

Almost immediately after Karmarkar's work appeared, researchers began to explore extensions of interior point methods to general convex optimization problems. Indeed, the nonlinear nature of interior point methods naturally suggested that such extensions were possible. Throughout the 1980s, a number of papers were written that showed that central trajectory methods and potential reduction methods for *LP* could be generalized to certain types of convex programs with theoretical performance guarantees, under a variety of restrictions (such as smoothness conditions) on the convex functions involved. However, there was no unifying theory or analysis. Then, in an incredible *tour de force*, Nesterov and Nemirovskii [15] presented a deep and unified theory of interior point methods for all of convex programming based on the notion of *self-concordant functions*. The intellectual contributions of this one research monograph cannot be overstated, whether it be for its mathematical depth, its implications for the theory of convex optimization and computational complexity, or for its implications for computation. To outline the thrust of this work, consider the following general convex program:

$$\begin{aligned} \text{CP:} \quad \text{ minimize} \quad & f(x) \\ \text{s.t.} \quad & g_i(x) \leq 0, \quad i=1, \dots, m, \end{aligned}$$

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where $g_i(x)$ is convex, $i=1, \dots, m$, and the objective function $f(x)$ is linear (if not, add a new variable t and a new constraint $f(x) \leq t$ and declare the new objective function to be "minimize t "). Let $D = \{x \mid g_i(x) < 0, i=1, \dots, m\}$, and suppose we have a (convex) barrier function $B(x)$ that goes to infinity as x goes to the boundary of D . Then the barrier problem associated with CP is:

$$\begin{aligned} BCP(\mu): \quad & \text{minimize} && f(x) + \mu B(x) \\ & \text{s.t.} && x \in D, \end{aligned}$$

and the central trajectory of $BCP(\mu)$ is the set of optimal solutions $x(\mu)$ to $BCP(\mu)$ parameterized by the barrier parameter μ .

Nesterov and Nemirovskii show that Newton's method is a very efficient tool for solving CP by tracing the central trajectory of $BCP(\mu)$, when the barrier function $B(x)$ has the property of *self-concordance*. Their approach is very general: $B(x)$ does not necessarily depend on the way the functions $g_i(x)$ are expressed. It just depends on the interior of the underlying feasible region. One of the central ideas in the understanding of self-concordance (and the interior point algorithms derived from them) is the use of the Hessian of $B(x)$ to induce a local norm at x . Let $H(x)$ be the Hessian of $B(x)$ at x . The *induced norm* at x is defined to be $n_x(v) = \sqrt{v^T H(x) v}$, which is a quadratic norm using the Hessian $H(x)$ as the quadratic form. Roughly speaking, a function $B(x)$ is a ϑ -self-concordant barrier function with barrier parameter ϑ if $B(x)$ satisfies the following conditions: (i) local changes in the Hessian of $B(\cdot)$ at two points x and y can be bounded by the induced norm at x of $(x-y)$, and (ii) the induced norm of the Newton step at x is no larger than $\sqrt{\vartheta}$. (Stated more colloquially, a function is a ϑ -self-concordant barrier if the Hessian of the function is a relevant tool in bounding the size of the Newton step (by $\sqrt{\vartheta}$) and in measuring changes in the Hessian itself.) Nesterov and Nemirovskii show that when a convex program has a ϑ -self-concordant barrier, then Newton's method improves the accuracy of a given solution of (CP) by at least t digits in $O(\sqrt{\vartheta}t)$ Newton steps.

At present, ϑ -self-concordant barriers are known for only a few, but very important, classes of convex programs. These include linear and convex quadratically constrained programs (where $B(x) = -\sum_{i=1}^m \ln(b_i - a_i^T x - x^T Q_i x)$ and $\vartheta = m$) and semidefinite programming (where $B(x) = -\ln(\det(X))$ for the $n \times n$ matrix X and $\vartheta = n$), as well as convex programs involving the second-order cone $\{x \mid x^T Q x \leq (c^T x + d)^2, c^T x + d \geq 0\}$, and even epigraphs of matrix norms. However, at least in theory, self-concordant barriers can be used to process *any* convex program efficiently: indeed, Nesterov and Nemirovskii show that every open convex set in \mathbb{R}^n possesses a ϑ -self-concordant barrier where $\vartheta \leq Cn$ for some universal constant C . The implications of this truly far-reaching result for the complexity of convex programming is now being explored.

Nesterov and Nemirovskii also provide a "barrier calculus" consisting of many simple tools which allow the derivation of self-concordant barriers for complicated convex sets, based on self-concordant barriers for simpler convex sets.

In addition, Nesterov and Nemirovskii also work on the following conic form of convex optimization:

$$\begin{aligned} KP: \quad & \text{minimize} && c^T x \\ & \text{s.t.} && Ax = b \\ & && x \in K, \end{aligned}$$

where K is a pointed, closed, convex cone with non-empty interior which possesses a ϑ -self-concordant barrier; their algorithms and performance guarantees apply easily to this case. This elegant form allows for better presentation and also makes it easier to draw parallels (when applicable) among interesting and well studied special cases of CP and KP, such as linear programming (where K is the nonnegative orthant) and semidefinite programming (where K is the cone of symmetric positive semidefinite matrices).

Finally, researchers such as Güler [7] are demonstrating deep connections between the theory of interior point methods using ϑ -self-concordant barriers, and other branches of mathematics including algebra, complex analysis, and partial differential equations.

At present, computational experience with interior point methods for convex programming is rather limited, except as noted in the case of semidefinite programming. However, researchers are optimistic that at least some of the success of interior point methods for linear and semidefinite programming will be realized for more general convex programs.

Homogeneous and Self-Dual Methods

A linear programming problem is called *self-dual* if its dual problem is equivalent to the primal problem. Given a linear program P and an initial (possibly infeasible) point (x^0, y^0, s^0) for which $x^0 > 0$ and $s^0 > 0$, a homogeneous and self-dual interior point method constructs the following artificial linear program $HSDP$ which is self-dual and almost homogeneous:

$$\begin{aligned} HSDP: \quad & \text{minimize} && ((x^0)^T s^0 + 1)\theta \\ & \text{s.t.} && Ax - b\tau + \bar{b}\theta = 0, \\ & && -A^T y + c\tau - \bar{c}\theta \geq 0, \\ & && b^T y - \bar{c}^T x + z\theta \geq 0, \\ & && -\bar{b}^T y + \bar{c}^T x - z\tau = -(x^0)^T s^0 - 1, \\ & && x \geq 0, \quad \tau \geq 0, \end{aligned}$$

where

$$\bar{b} = b - Ax^0, \quad \bar{c} = c - A^T y^0 - s^0, \quad \bar{z} = c^T x^0 + 1 - b^T y^0.$$

It is not hard to see that this program is self-dual, because the coefficient matrix is skew-symmetric. Denote the slacks on the second and third set of constraints by s and κ . Then $HSDP$ has a trivial feasible interior point $(x, \tau, \theta, y, s, \kappa) = (x^0, 1, 1, y^0, s^0, 1)$ that can be used to initiate any interior point algorithm for solving $HSDP$. Since the dual is equivalent to the primal, the optimal value of $HSDP$ is zero. By using a path following interior point algorithm, one can compute a strictly self-complementary solution $(x^*, \tau^*, \theta^*, y^*, s^*, \kappa^*)$ such that $\theta^* = 0, x^* + s^* > 0$, and $\tau^* + \kappa^* > 0$. If $\tau^* > 0$, then

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$x^* \tau^*$ is an optimal solution of P and $(y^* / \tau^*, s^* / \tau^*)$ is an optimal solution of D . Otherwise, by strict complementarity, $\kappa^* > 0$, whereby from the third constraint it follows that either $c^T x^* < 0$ or $-b^T y^* < 0$. The former case implies the infeasibility of the primal problem P , and the latter case implies the infeasibility of the dual problem D .

The homogeneous and self-dual interior point method possesses the following nice features: (i) it solves a linear program P without any assumption concerning the existence of feasible, interior feasible, or optimal solutions, (ii) it can start at any initial point, feasible or not, (iii) each iteration solves a system of linear equations whose size is almost the same as for standard interior-point algorithms, (iv) if P is feasible, it generates a sequence of iterates that approach feasibility and optimality simultaneously, otherwise it correctly detects infeasibility for at least one of the primal and dual problems, and (v) it solves a problem in polynomial time ($O(\sqrt{n}L)$ iterations) without using any "big M " constants. We point out that the infeasible interior point algorithms presented in the second section do not possess this last feature. Homogeneous and self-dual interior point methods have the promise to be competitive in practice with standard interior point software, see Xu et al. [23].

Homogeneous and self-dual methods can be extended to more general problems, such as linear and nonlinear complementarity problems. We refer readers to Ye, et al. [24] for an initial description.

Linear and Nonlinear Complementarity Problems

The standard linear complementarity problem, or LCP , is to find a pair (x, s) of n -dimensional variables that satisfy the linear constraint

$$s = Mx + q$$

and the complementarity conditions

$$(x, s) \geq 0, \quad x_j s_j = 0, \quad j=1, \dots, n,$$

where M is an $n \times n$ matrix and $q \in \mathbb{R}^n$. The optimality conditions for both linear programming and convex quadratic programming can be cast as an instance of LCP , and for this reason LCP is often used as a general model in the development and analysis of interior-point algorithms.

While there are several important classes of the LCP , the most important class is the *monotone LCP*, defined to be those instances for which the set of feasible solutions (x, s) are maximal and monotone in \mathbb{R}^{2n} (equivalently, for which the matrix M is positive semidefinite). Linear and convex quadratic programming problems fall into this class. More generally, instances of LCP are typically classified according to classes of the matrix M , such as P_0 -matrices and $P_*(\kappa)$ -matrices (see Kojima et al. [11] for definitions).

Interior point methods for solving LCP have been developed using the following generalization of the central trajectory equation system (2):

$$\begin{aligned} s &= Mx + q, \quad x > 0, \quad s > 0, \\ XSe - \mu e &= 0. \end{aligned} \quad (7)$$

If the matrix M is a P_0 -matrix and a feasible interior point exists, then the set of solutions to (7) forms a path (central trajectory) parameterized by μ , leading to a solution of LCP as μ goes to 0, and so one can solve the standard LCP with a P_0 matrix by using a path-following interior point algorithm. This approach extends to infeasible interior point methods, and potential reduction methods for solving LCP have also been proposed by researchers. In the case of the monotone LCP , many interior point algorithms have a polynomial time performance guarantee. For $P_*(\kappa)$ -matrix LCP , there is also an explicit complexity analysis and performance guarantee. The solution of LCP with P_0 -matrices is known to be NP-complete.

The nonlinear complementarity problem, or $NLCP$, is the problem for finding (x, s) such that

$$s = f(x), \quad (x, s) \geq 0, \quad XSe = 0,$$

for a given continuous function $f(\cdot)$. If $f(\cdot)$ is monotone, $NLCP$ is also called monotone. The optimality conditions for convex constrained optimization can be cast as an instance of the monotone $NLCP$. For this class of $NLCP$, the central trajectory system (7) can be suitably generalized, and so can be solved by path-following interior point methods. Interior point methods for more general classes of $NLCP$ are discussed in Kojima et al. [12].

Recently researchers have become interested in the semidefinite complementarity problem, or $SDCP$, which is a special case of $NLCP$ arising in the study of semidefinite programming (see the earlier section). Infeasible interior point algorithms have been developed for the monotone instances of $SDCP$, and $SDCP$ is currently a very active research problem.

Some Theoretical Issues Related to Interior Point Methods

Recently, theoretical research on the complexity of solving linear programming has focused on developing appropriate measures for adequately representing the "size" of an LP instance that are more relevant to computation than traditional measures of "size" such as the dimensions m and n or the bit-size L of a binary representation of an LP instance. In this closing section, we discuss two such measures, namely $C(d)$ of an LP data instance $d=(A, b, c)$, and $\bar{\chi}_A$ for the matrix A .

Consider the very general convex optimization problem cast as follows:

$$\begin{aligned} P(d): \quad & \text{maximize} \quad c^T x \\ & \text{s.t.} \quad b - Ax \in C_y \\ & \quad \quad x \in C_x, \end{aligned}$$

where C_x and C_y are closed convex cones, and the data d for the problem is the array $d=(A, b, c)$. Any convex optimization problem can be cast in this form, including LP as a special case. The terminology $P(d)$ emphasizes the dependence of the problem on the data $d=(A, b, c)$. Renegar [16] develops a *condition number* $C(d)$ for $P(d)$ that is intuitively appealing, arises naturally in considering the problem $P(d)$, is an extension of the traditional condition number for systems of linear equations, and possesses many attractive

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geometric and algebraic characteristics. (For example, if $P(d)$ has a feasible solution, then it must have a feasible solution whose norm is no larger than $C(d)$.) We give a rough description of $C(d)$ as follows. Let $d=(A,b,c)$ be the data for $P(d)$ and let $\Delta d = (\Delta A, \Delta b, \Delta c)$ be a change in the data. Let $\rho(d)$ be the smallest change Δd needed to make the problem $P(d)$ either infeasible or unbounded. Then $C(d)$ is defined to be $\|d\|/\rho(d)$. That is, $C(d)$ is a scale-invariant version of the reciprocal of the smallest change in the data $d=(A,b,c)$ needed to cause $P(d+\Delta d)$ to be ill-behaved. Roughly speaking, Renegar shows that the complexity of an interior point method for solving $P(d)$ is inherently sensitive only to the condition number $C(d)$ of the underlying problem and to the barrier parameter ϑ of the self-concordant barrier for the cones C_x and C_y , and that the complexity bound on the number of iterations is $O(\vartheta(\ln(C(d) + \ln(1/\epsilon)))$ to produce an ϵ -optimal solution of $P(d)$. Therefore, the interior point algorithm is efficient in a well-defined sense. Not surprisingly, the condition number $C(d)$ is intrinsically involved in a variety of special properties of the central trajectory of a linear program (see Freund and Nunez [5]), and we anticipate that the study of the condition number $C(d)$ will yield much new insight into linear and convex programming in the future.

Another very interesting development due to Vavasis and Ye [21] is an interior point algorithm for linear programming whose running time depends only on the dimension n and on a certain measure of the matrix A denoted by $\bar{\chi}_A$. Let (A,b,c) be the data for an LP instance, where the data are *not* restricted to be rational numbers. For the matrix A , define the quantity:

$$\bar{\chi}_A = \sup\{\|A^T(ADA^T)^{-1}AD\| \mid D \text{ is a positive } n \times n \text{ diagonal matrix}\}.$$

Then Vavasis and Ye present an interior point algorithm for solving a linear program in at most $O(n^{3.5}(\ln(\bar{\chi}_A) + \ln(n) + C))$ iterations of Newton's method, where C is a universal constant. The significance of this result derives from the fact that the data b and c play no role in the bound on the number of iterations. Put a different way, the efficiency of their algorithm for linear programming depends only on the dimension and on a certain algebraic property of the matrix A embodied in the quantity $\bar{\chi}_A$. This research improves on earlier work by Tardos, by showing that the dependency of the complexity result on A is true even when the data are not presumed to be integer (or rational) coefficients.

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References

- [1] F. Alizadeh, Interior point methods in semidefinite programming with applications to combinatorial optimization, *SIAM Journal on Optimization* 5 13-51, 1995.
- [2] F. Alizadeh, J.P. Haeberly, and M. Overton, Primal-dual interior-point methods for semidefinite programming: convergence rates, stability and numerical results, New York University Computer Science Dept Report 721, May 1996.
- [3] E. Andersen, J. Gondzio, C. Meszaros, and X. Xu, Implementation of interior point methods for large scale linear programming, in *Interior point methods in mathematical programming*, T. Terlaky, ed., Kluwer Academic Publisher, 1996.
- [4] K. Anstreicher, Potential Reduction Methods, in *Interior point methods in mathematical programming*, T. Terlaky, ed., Kluwer Academic Publishers, 1996.
- [5] R.M. Freund and M. Nunez, Condition measures and properties of the central trajectory of a linear program, submitted to *Mathematical Programming*, M.I.T. Sloan School working paper 3889-96 MSA, March, 1996.
- [6] C. Gonzaga, Path following methods for linear programming, *SIAM Review* 34(2) 167-227, 1992.
- [7] O. Güler, Barrier functions in interior point methods, *Mathematics of Operations Research* 21 (1996), to appear.
- [8] D. den Hertog, *Interior point approach to linear, quadratic and convex programming, algorithms, and Complexity*, Kluwer Publishers, Dordrecht, The Netherlands, 1994.
- [9] B. Jansen, C. Roos, and T. Terlaky, A short survey on ten years of interior point methods, Report 95-45, Delft University of Technology, Delft, The Netherlands, 1995.
- [10] N. Karmarkar, A new polynomial-time algorithm for linear programming, *Combinatorica* 4 373-395, 1984.
- [11] M. Kojima, N. Megiddo, T. Noma, and A. Yoshise, *A unified approach to interior point algorithms for linear complementarity problems*, Lecture Notes in Computer Science 538, Springer-Verlag, Berlin (1991).
- [12] M. Kojima, N. Megiddo, and S. Mizuno, A general framework of continuation methods for complementarity problem, *Mathematics of Operations Research* 18 945-963, 1994.
- [13] Lustig, R. Marsten, and D. Shanno, Interior point methods: computational state of the art, *ORSA Journal on Computing* 6 1-14, 1994.
- [14] S. Mizuno, Infeasible-interior-point algorithms, in *Interior point methods in mathematical programming*, T. Terlaky, ed., Kluwer Academic Publishers, 1996.
- [15] Y. Nesterov and A. Nemirovskii, *Interior-point polynomial algorithms in convex programming*, SIAM Publications, Philadelphia, U.S.A., 1994.
- [16] J. Renegar, Linear programming, complexity theory and elementary functional analysis, *Mathematical Programming* 70, 279-351, 1995.
- [17] R. Saigal, *Linear Programming: A Modern Integrated Analysis*, Kluwer Academic Publishers, Boston, 1995.
- [18] M. Todd, Potential reduction methods in mathematical programming, *Mathematical Programming B*, to appear.
- [19] T. Tsuchiya, Affine scaling algorithm, in *Interior point methods in mathematical programming*, T. Terlaky, ed., Kluwer Academic Publishers, 1996.
- [20] L. Vandenberghe and S. Boyd, Semidefinite programming, *SIAM Review* 38 1 49-95, 1996.
- [21] S. Vavasis and Y. Ye, A primal-dual interior point method whose running time depends only on the constraint matrix, *Mathematical Programming* 74 79-120, 1996.
- [22] S. Wright, *Primal-Dual Interior Point Algorithms*, SIAM Publications, Philadelphia, forthcoming.
- [23] X. Xu, P.-F. Hung, and Y. Ye, A simplified homogeneous and self-dual linear programming algorithm and its implementation, *Annals of Operations Research* 62, 151-171, 1996.
- [24] Y. Ye, M. Todd, and S. Mizuno, An $O(\sqrt{nL})$ -iteration homogeneous and self-dual linear programming algorithm, *Mathematics of Operations Research* 19 53-67, 1994.

CONFERENCE

► **Second International Symposium on Operations Research and its Applications (ISORA '96)**
Guilin, China
Dec. 11-13, 1996

► **Optimal Control: theory, algorithms, and applications**
Center for Applied Optimization
University of Florida
Feb. 27 - March 1, 1997

► **Third Workshop on Models and Algorithms for Planning and Scheduling Problems**
Cambridge, England
April 7-11, 1997

► **XVI International Symposium on Mathematical Programming, Lausanne**
Switzerland, Aug. 1997

► **ICM98**
Berlin, Germany
August 18-27, 1998

Report on the Fifth International Symposium on Generalized Convexity
June 17-21, 1996, Centre International de Rencontres Mathematiques (CIRM), Marseille-Luminy

The Fifth International Symposium on Generalized Convexity was held at Centre International de Rencontres Mathematiques (CIRM) in Marseille-Luminy, June 17-21, 1996. Among others, it was sponsored by the Mathematical Programming Society and the International Scientific Committee for the recently founded Working Group on Generalized Convexity.

About 70 scholars from 15 countries followed a tight schedule of more than 50 talks. The papers covered many aspects in the field such as fractional programming, various kinds of generalized convex sets and functions, optimality conditions and duality, generalized monotonicity in variational inequalities and equilibrium problems, multiobjective programming, nondifferentiable calculus. An emphasis on the fast developing theory of generalized monotone operators was noticeable. The bulk of the contributions on generalized convexity were of a theoretical nature. In addition, several authors presented algorithmic results of applications, e.g., in economics and mechanics.

Especially well received were invited lectures by F. Clarke, J.S. Pang and R. Wets in which generalized convexity was related to topics in neighboring fields such as nonsmooth analysis, mathematical programming/complementarity theory/variational inequalities and stochastic programming.

The conference site (CIRM) provided excellent opportunity for limitless professional contact among the participants. A tour to nearby Aix-en-Provence and a taste of its world-famous bouillabaisse rounded off a rich scientific program.

A selection of refereed papers presented at the symposium will appear in the Proceedings to be published by Kluwer Academic Publishers. For information, also on the proceedings of the previous four symposia in Vancouver (1980), Canton (1986), Pisa (1988) and Pecs (1992), please contact one of the undersigned. The sixth symposium has been scheduled to take place on the Greek Island of Samos in 1999.

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ICM98

Berlin, Germany
August 18-27, 1998

**Nominations for the Fields
 Medals and the Rolf
 Nevanlinna Prize**

The Fields Medals and the Rolf Nevanlinna Prize will be presented at the opening ceremony of ICM98 on August 18, 1998. The International Mathematical Union (IMU) Executive Committee has appointed a Fields Medals and a Nevanlinna Prize Committee to select the awardees.

An individual may contribute to the selection process by contacting the Committee of Mathematics of his or her country. (To find your National Mathematical Committee, look up the list of IMU member countries in the IMU-server (<http://elib.zib-berlin.de/IMU>) and click on your own country.) Click on "Fields Medals and Nevanlinna Prizes" for additional information.

The National Committees may suggest candidates. The nominations must be accompanied by a brief justification and be received no later than March 31, 1997 by the Secretary of IMU:

International Mathematical Union
 Professor Jacob Palis, Secretary
 Estrada Dona Castorina, 110, Jardim
 Botânico 22.460-320 - Rio de
 Janeiro, RJ Brazil
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 e-mail: IMU@IMPA.BR

More information about
 ICM98 may be found in the
 ICM98 WWW-server
 (URL: <http://elib.zib-berlin.de/ICM98>). This WWW-server

also offers an electronic preregistration form for ICM98. If you do not have access to the World Wide Web and would like to subscribe to the ICM98 circular letters, just send an e-mail to icm98@zib-berlin.de and put "preliminary preregistration" in the subject line.

The IMU plans to award travel grants to young mathematicians to help them attend the ICM98. These grants are intended primarily for young mathematicians, up to 35 years of age, from developing countries. Candidates should give their name and address and include evidence of research at the post-doctoral level in addition to a brief curriculum vitae showing date of birth plus a list of publications. The Local Organizing Committee of ICM will provide a special allowance to the grantees to cover registration, board and lodging.

Applications for travel grants may be sent directly to the Secretary of the Union or may be submitted through the Committees for Mathematics. All applications should reach the Secretary of IMU by January 1, 1998 (address listed above).

IMU would like to increase the number of travel grants and is seeking donations to its Special Development Fund (SDF) for this purpose. Donations can be sent at any time and in any convertible currency to the following account: IMU Account Number 0862-656208-21, Schweizerische Kreditanstalt, Stadtfiliale Zurich - Rigiplatz, Universitaetstrasse 105, CH-8033 - Zurich - Switzerland.

-Martin Gröetschel
martin.groetschel@zib-berlin.de

**Third Workshop on Models and Algorithms for Planning and
 Scheduling Problems**
Cambridge, England
April 7-11, 1997

Following two successful workshops at Lake Como, Italy, in 1993, and in Wernigerode, Germany, in 1995, the Third Workshop on Models and Algorithms for Planning and Scheduling Problems is to be held in Cambridge, England, April 7-11, 1997.

The workshop aims to provide a forum for scientific exchange and cooperation in the field of planning, scheduling and related areas. To maintain the informality of the previous workshops and to encourage discussion and cooperation, there will be a limit of 80 participants and a single stream of presentations. Contributions on any aspect of scheduling are welcome.

Authors presenting papers at the workshop are invited to submit their manuscripts for possible publication in a special issue of *Annals of Operations Research*. Papers for this special issue will be refereed in the normal way.

Invited Speakers

Y.J. Crama, University of Liege
 K.D. Glazebrook, Univ. of Newcastle
 D.S. Johnson, AT&T Bell Laboratories
 J.K. Lenstra, Tech. Univ. of Eindhoven
 D.B. Shmoys, Cornell University

Organizers

M.A. H. Dempster, Cambridge
 C.A. Glass, Southampton
 C.N. Potts, Southampton
 V.A. Stusevich, Greenwich
 F. Vanderbeck, Cambridge
 R.R. Weber, Cambridge

Persons interested in participating should submit a pre-registration form and, if relevant, an abstract of the intended contribution before Dec. 20, 1996.

A copy of the pre-registration form and details about dates, accommodation, fees, travel, etc., can be obtained by viewing

<http://www.statslab.cam.ac.uk/workshop/>

or by sending your postal address and a request for a printed workshop brochure to secretary@statslab.cam.ac.uk.

Symposium News

The number of preregistrations so far for the 16th International Symposium on Mathematical Programming, to be held in Lausanne, August 24-29, 1997, is now almost 600! The number of invited sessions is also growing fast and is now 117. Next to the parallel sessions there will be more than 20 semi-plenary lectures featuring the state of the art in mathematical programming. Other special events include a SIMPLEX METHOD 50th BIRTHDAY PARTY, and work is underway to establish a new prize for any progress in developing a polynomial time *simplex* algorithm for linear programming problems or giving a proof that no such algorithm exists. Important news for people from former Communist countries in Eastern Europe is that they can apply for financial aid at their local SOROS foundation. The Symposium Chairman has been told that such applications will be granted. The symposium has a home page at the URL <http://dmawww.epfl.ch/roso.mosaic/ismp97>. Here you can find the latest symposium news, preregistration forms, and an overview of all sessions organized so far. If you have not preregistered, please do so quickly so that you do not miss important information. More comprehensive news will follow in the next issue of OPTIMA.

On behalf of the Symposium Organizers, KAREN AARDAL

D.R. Fulkerson Prize Call for Nominations

This is a call for nominations for the D. Ray Fulkerson Prize in Discrete Mathematics of the American Mathematical Society and the Mathematical Programming Society. The Prize will be awarded at the XVth International Symposium on Mathematical Programming to be held in Lausanne, Switzerland, August 24-29, 1997.

The specifications for the Fulkerson Prize read: "Papers to be eligible for the Fulkerson Prize should have been published in a recognized journal during the six calendar years preceding the year of the Congress. The extended period is in recognition of the fact that the value of fundamental work cannot always be immediately assessed. The prizes will be given for single papers, not series of papers or books, and in the event of joint authorship the prize will be divided. The term "Discrete Mathematics" is intended to include graph theory, networks, mathematical programming, applied combinatorics, and related subjects. While research work in these areas is usually not far removed from practical applications, the judging of papers will be based on their mathematical quality and significance." So papers eligible for the 1997 prize should have appeared in one of the six years 1991–1996. The conclusions of the Fulkerson Prize Committee (Éva Tardos chair, Ron Graham, and Ravi Kannan) will be presented to the two Societies jointly awarding the Prize: the Mathematical Programming Society and the American Mathematical Society.

Please send your nominations (including reference to the nominated article and some evaluation) by January 30, 1997 to the chair of the Fulkerson Prize Committee:

Éva Tardos, Department of Computer Science, Cornell University, Ithaca, NY 14850
e-mail: eva@cs.cornell.edu.

OR-Library Updated

The OR-Library (o.rlibrary@ic.ac.uk) now contains the following problem types:

- multiple depot vehicle routing problem - see the file *multivrpinfo*
 - network flow problem - see the file *nefflowinfo*
 - p-hub location problem - see the file *phubinfo*
 - period travelling salesman problem - see the file *periodtspinfo*
 - period vehicle routing problem - see the file *periodinfo*
 - set covering problem - see the file *scpinfo*
 - timetabling problem - see the file *tableinfo*
- ftp access available at mscmga.ms.ic.ac.uk
WWW access available at <http://mscmga.ms.ic.ac.uk/>

Update of Mathematics Subject Classification

The Mathematical Programming Society has set up a committee consisting of Jose Mario Martinez (martinez@ime.unicamp.br), Lex Schrijver (Lex.Schrijver@cw.nl), and Mike Todd (miketodd@CS.Cornell.EDU), chair, to provide input to *Mathematical Reviews* and *Zentralblatt für Mathematik* on their proposed update of the 1991 Mathematics Subject Classification. It is not envisioned that this will be a major update, but it needs to incorporate new areas of research and possibly appropriately reorganize existing areas. The committee expects to concentrate its attention on the organization of 90CXX, mathematical programming but may also make suggestions on other parts of the classification, such as 65KXX, 68QXX, 05DXX, and 90BXX.

The existing classification can be found on the world wide web at URL <http://www.ams.org/msc/>, or copies may be obtained by e-mail or regular mail from the committee members.

We welcome suggestions from the mathematical programming community which can be sent by e-mail to any or all of the committee members or by regular mail to the address below. We would appreciate receiving input before or during the International Symposium on Mathematical Programming to be held in Lausanne, Switzerland, Aug. 1997.

Michael J. Todd
School of Operations Research and Industrial Engineering
Rhodes Hall
Cornell University
Ithaca, NY 14853-3801 USA

SIAM Announces Postgraduate Memberships

SIAM is pleased to announce a new class of membership to be offered in 1997 to current and new members who are recent graduates. "Postgraduate Memberships" are available on a one-time basis for up to three consecutive years immediately after receiving your highest degree. Dues for 1997 are \$45.

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Call for Nominations for the George B. Dantzig Prize

Nominations are solicited for the George B. Dantzig Prize, administered jointly by the Mathematical Programming Society (MPS) and the Society for Industrial and Applied Mathematics (SIAM). This prize is awarded to one or more individuals for original research which, by virtue of its originality, breadth and depth, is having a major impact on the field of mathematical programming. The contributions eligible for consideration must be publicly available and may address any aspect of mathematical programming in its broadest sense. Strong preference is given to contributions by individuals under 50 years of age. The prize will be presented at the symposium. Past Dantzig Prize recipients have been: M.J.D. Powell and R.T. Rockafellar in 1982, E.L. Johnson and M.W. Padberg in 1985, M.J. Todd in 1988, M. Gröetschel and A.S. Nemirovsky in 1991, and C. Lemaréchal and R. J.-B. Wets in 1994. The prize committee members are: Ellis L. Johnson, Chairman; Jorge Moré; Claude Lemaréchal and Clovis Gonzaga. Please send nominations to Ellis L. Johnson, Georgia Institute of Technology, School of Industrial and Systems Engineering, Atlanta, Georgia 30332-0205, U.S.A., or by e-mail to ejohnson@isye.gatech.edu. Nominations are due by November 1, 1996, and should provide a brief one- or two-page description of the nominee's outstanding contributions, and, if possible, a current resume including a list of the nominee's publications.

- Ellis Johnson

Announcements

Global Minimization of Nonconvex Energy Functions: Molecular Conformation and Protein Folding

P.M. Pardalos, D. Shalloway
and G. Xue (Eds.)

DIMACS Series in Discrete Mathematics
and Theoretical Computer Science, Vol. 23,
Providence, R.I.

American Mathematical Society

ISBN NUMBER 0-8218-0471-5

The book is a collection of 17 papers entirely devoted to the problem of protein folding, which can be described as follows: Given the amino acid sequence of a protein, find the protein three-dimensional conformation (fold-

ing) in a bio-chemical environment. The basic approach of the entire collection is that the sought folding must minimize an appropriate energy function, and this requires a concerted effort of the specialists in biochemistry (to produce an adequate energy function), in optimization (to suggest a minimization method), and in computer science (to create a reliable and friendly computational environment in a problem involving a huge number of variables and constraints).

The book is a set of refereed papers based on talks presented at a workshop held at DIMACS (an NSF Science and Technology Center located on a campus of Rutgers University, NJ), March 1995. The arrangement is only by alphabetical order and no index is provided.

The collection can be partitioned into three classes: (1) Adaptation of global optimization methods to the energy functions specified, sometimes with much discussion and substantiation of the physics (9 papers: P. Amara et al., B.W. Church et al., J. Kostrowicki and H.A. Scheraga, C.D. Maranas et al., R. Pachter et al., A.T. Phillips et al., G. Ramachandran and T. Schlick, G.L. Xue et al., and M.M. Zacharias and D.G. Vlachos), (2) Computer implementations based on specifics on the problems (4 papers: R.E. Brucoleri, R.H. Byrd, J. Gu and B. Du, and X. Hu et al.), (3) Associated problems (4 papers: H.A. Hauptman, J.J. Moré and Z. Wu, A. Sali et al., and M. Vieth et al.).

Among the methods discussed, there are such techniques as simulated annealing (P. Amara, J. Ma, and J.E. Straub; B.W. Church, M. Oresic, and D. Shalloway; R. Pachter, Z. Wang, J.A. Lupo, S.B. Fairchild, and B. Sennett; and M.M. Zacharias and D.G. Vlachos) and local search (R.E.

Brucoleri; R.H. Bird, E. Eskow, A. van der Hoek, R.B. Schnabel, C.-S. Shao, and Z. Zou; J. Gu and B. Du; and A.T. Phillips, J.B. Rosen, and V.H. Walke). It should be pointed out that the techniques here are not (and perhaps cannot be) applied as they are but in a much enhanced form and combined with some other useful techniques (including branch and bound search, smoothing, convex optimization, etc.). In this aspect, an original and perhaps most promising direction seems to be the one which involves aggregation of the original model (usually, in terms of its local solutions) according to the "cluster" structure of the folding, with subsequent performing optimization for the aggregate model. This idea is employed in many papers, especially in those of R.H. Bird, E. Eskow, A. van der Hoek, R.B. Schnabel, C.-S. Shao, and Z. Zou; B.W. Church, M. Oresic, and D. Shalloway; and G.L. Xue, A.J. Zall, and P.M. Pardalos.

The level of presentation and the potential background of the reader are quite different for the different papers. However, there are several articles oriented toward a mathematician having a quite rudimentary knowledge of molecular biology, with careful motivation of the presented models and methods. Among them, the reviewer would recommend the papers of C.D. Maranas, I.P. Androulakis, and C.A. Floudas; R.H. Bird, E. Eskow, A. van der Hoek, R.B. Schnabel, C.-S. Shao, and Z. Zou; J.J. Moré and Z. Wu; G.L. Xue, A.J. Zall, and P.M. Pardalos; and J. Kostrowicki and H.A. Scheraga.

Overall, the book can be considered as an important step in consolidation of the newly emerging discipline which, currently, seems quite a multidisciplinary one.

—BORIS MIRKIN AND ILYA MUCHNIK

Global Optimization in Action

J.D. Pinter

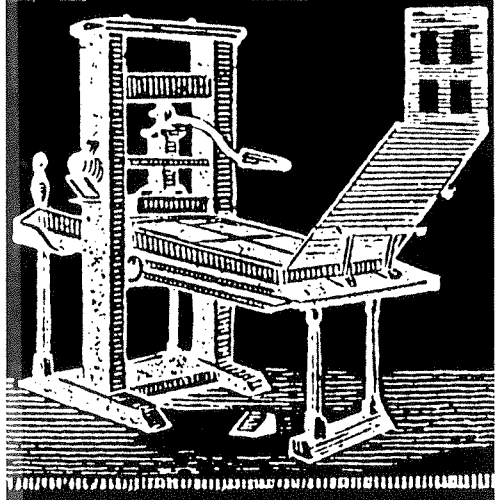
Nonconvex Optimization and Its
Applications, Series 6, Kluwer Academic
Publishers, Dordrecht 1996

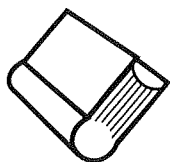
ISBN NUMBER 0-7923-3757-3

It is hard to understand why nearly all the research in nonlinear optimization deals with the search for local solutions. Is it a more beautiful theory, is it the narrower connections to classical mathematics like analysis or geometry, is it the lower costs for numerical computations, or is it that, in practice, the expert's knowledge combined with the control by the human brain enables the user to concentrate the search for such local solutions that are

BOOK

REVIEW





almost global ones? Why have there not been published more than two dozen monographs about the global variant? No matter what the reasons are, Pinter's book is an important contribution to make the theoretical and practical fundamentals of multiextremal optimization more accessible and to spread the global view of optimization.

The goal of this well-written monograph is to introduce the reader to deterministic, gradient-free optimization algorithms which are based on adaptive, sequential partitions and search in the feasible set. Besides the deterministic methodology interweaving the topics of the book, a shorter discussion of adaptive stochastic algorithms is included. The core of the book, however, is a broad spectrum of thoroughly discussed and systematically presented applications, which originate mainly from joint research of the author with colleagues during the past decade.

The book is subdivided into four parts. Part 1 (Global Optimization: A Brief Review; 38 pages) can be seen as a first look at global optimization and several basic methods. Part 2 (Partition Strategies in Global Optimization: The Continuous and the Lipschitzian Case; 105 pages) contains the theoretical core of the book, that is, adaptive deterministic partition algorithms and their convergence theory. In particular, the partition topics cover intervals, boxes, simplices, general convex sets and star sets. Part 3 (Implementation Aspects, Algorithm Modifications and Stochastic Extensions; 100 pages) deals with several algorithmical implementations of Lipschitz functions and also with a few stochastic extensions such as decision making under uncertainty, random search based stochastic optimization procedures, and optimization with noise-perturbed function values. Part 4 (Applications, 184 pages) is the focal point of the book and is concerned with existing and prospective applications of continuous and Lipschitz global optimization. Among the many presented, one finds methods to solve nonlinear equations and inequalities, data classification, aggregation of methods to solve nonlinear equations and inequalities, data classification, aggregation of expert opinions, product design, wastewater treatment system design, multiple point source river pollution management, risk analysis and management of accidental toxic river pollution, etc.

Pinter's book can be recommended to each reader who needs a far-reaching survey of continuous and Lipschitz optimization. The book is distinguished by its vivid and informative style, the large number of areas that are touched, the lack of longwinded and overly sophisticated expositions, and the optimum balance between theory and practical applications.

—H. RATSCHKE

Linear Optimization and Extensions

Manfred Padberg

Series in Algorithms and Combinatorics 12
Springer, Berlin, 1995

ISBN NUMBER 3-540-5734-9

Linear optimization is at the core of operations research and therefore it is no wonder that there exist already several texts on this subject. However, the new book of Manfred Padberg is not one of the books summarizing and slightly modifying the contents of the previous ones but provides an approach to linear optimization that is strongly motivated by actually solving linear programs on a computer.

The book consists of ten chapters and three appendices. Chapters 1 through 6 cover the basics of linear optimization, the primal and the dual simplex algorithm, and duality theory. As Padberg's intent is the solution of large linear programs, he does not introduce the simplex algorithm with the help of the well-known tableau method, but uses only basic linear algebra. This approach gives the reader the exact knowledge of the simplex algorithm and what has to be actually done in an implementation of the simplex algorithm that solves problem instances larger than typical text book examples. Moreover, a dynamic simplex algorithm and important aspects of practical computations such as cycling and data structures are discussed.

Chapter 7 is a comprehensive outline of the analytical geometry, in particular polyhedral theory, related to linear optimization. The equivalence of the description of polyhedra by linear inequalities and by the convex and conical combinations of points is proven. But even in these more theoretical sections, computational aspects play a central role. The double description algorithm transforming any representation into the other is outlined. Implementations of this algorithm are a basic tool for the investigation of polyhedra associated with combinatorial optimization problems. The chapter contains also some aspects of complexity theory of linear optimization and of the geometry of the simplex algorithm.

The currently most important "competitors" of the simplex algorithm, interior point algorithms and their modifications, are the contents of chapter 8. Mainly projective algorithms are discussed, but also a description of barrier methods is included.

The ellipsoid method, described in Chapter 9, is mainly of theoretical interest due to the numerical problems arising in its implementation. However, the proof of the equivalence of optimization and separation is an important theoretical aspect of the linear optimization approach to integer optimization problems.

The practical application of linear optimization techniques in combinatorial optimization is briefly presented in the last chapter of the book. The solution of the mixed integer optimization problem by a branch-and-cut algorithm is shown and the description of the associated polyhedra is discussed.

The appendices give three examples of interesting applications of linear programming from financial management, from control of a refinery, and from the solution of traveling salesman problems. These examples are not "toy" examples often found in other books but real world examples motivating the reader to apply linear optimization.

The book contains a series of instructive exercises and can be recommended as a text for a course in linear programming for students with a sufficient knowledge of linear algebra. Since Padberg always has in mind the application of linear optimization to integer and combinatorial optimization, this book should be particularly helpful in a series of lectures on linear and mixed integer optimization.

Padberg's impressive book deals with solving linear optimization problems on modern computers. Therefore, this book is a useful reference book for everybody working in computational linear and integer programming.

Unfortunately, the bibliography can only be found at the end of the book and almost no references are cited within the text. A short annotated bibliography at the end of each section or chapter would have been helpful for the reader to delve more deeply into the topics of the book.

Reading this book is a real pleasure. Padberg not only teaches the theory and practice of linear optimization in a clearly structured way; he also is entertaining in motivating examples of linear modeling, in describing the impacts on linear optimization algorithms and in giving his detailed personal accounts of using and refining these algorithms. He even has stories from Greek mythology and not just about the achievements of early mathematicians.

—STEFAN THIENEL

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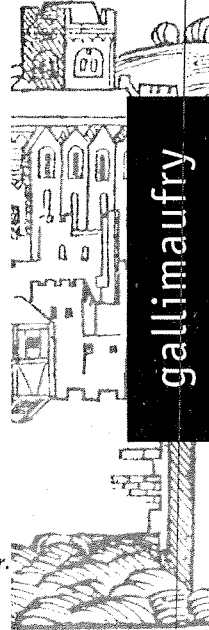
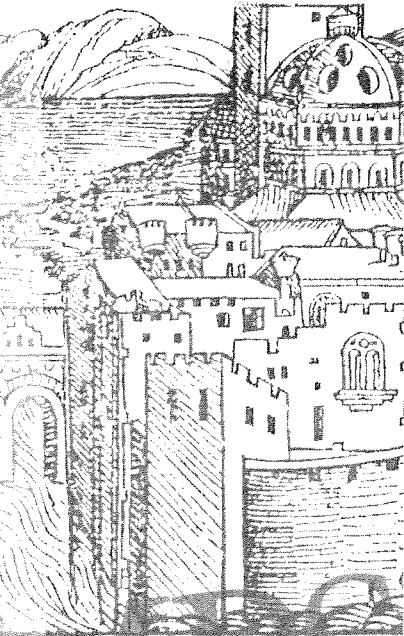
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■ Hiroshi Konno is Dean of the Graduate School of Design, Systems and Technology at the Tokyo Institute of Technology. ■ Achim Bachem formerly of U of Cologne has moved to the executive board of the German Aerospace Research Establishment (DLR). His email is bachem@dlr.de. ■ The Society now has a homepage at the URL <http://www.ccam.rice.edu/mathprog/> thanks mainly to the efforts of Steve Wright. On the page you can find information about the society and about becoming a member, some articles that have appeared in OPTIMA, information about our journal, *Mathematical Programming*, a pointer to the Symposium homepage and much more. If you are unfamiliar with the Web, you can find a tutorial by Samer Takriti in OPTIMA No. 47, 1995. Go surfing! ■ Deadline for the next OPTIMA is Nov. 15, 1996.

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