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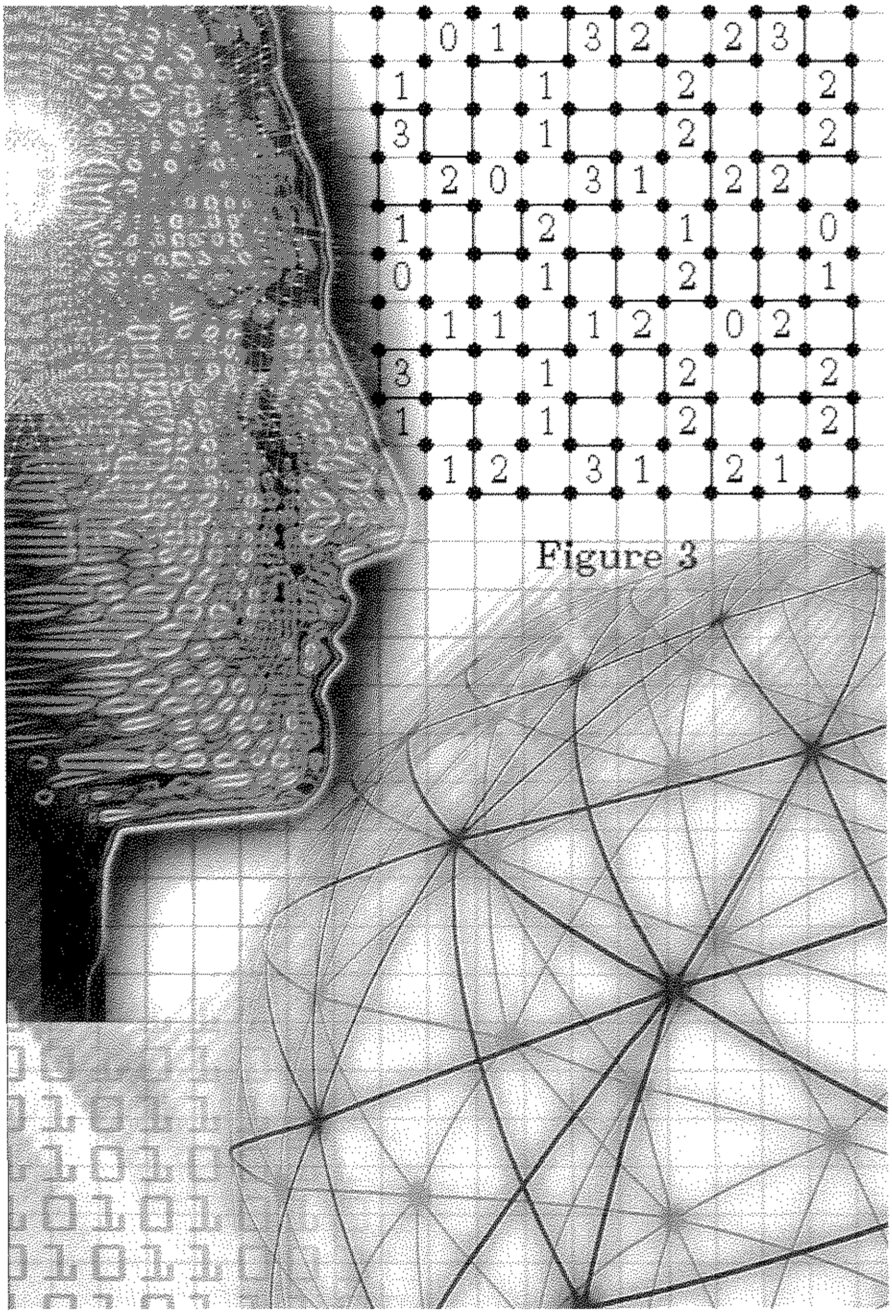

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The Strong Perfect Graph Theorem

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1. Introduction

In this note, all graphs are simple (no loops or multiple edges) and finite. The vertex set of graph G is denoted by $V(G)$ and its edge set by $E(G)$. A *stable set* is a set of vertices no two of which are adjacent. A *clique* is a set of vertices every pair of which are adjacent. The cardinality of a largest clique in graph G is denoted by $\omega(G)$. The cardinality of a largest stable set is denoted by $\alpha(G)$. A *k-coloring* is a partition of the vertices into k stable sets (these stable sets are called *color classes*). The *chromatic number* $\chi(G)$ is the smallest value of k for which there exists a k-coloring. Obviously, $\omega(G) \leq \chi(G)$ since the vertices of a clique must be in distinct color classes of a k-coloring. An *induced subgraph* of G is a graph with vertex set $S \subseteq V(G)$ and edge set comprising all the edges of G with both ends in S. It is denoted by $G(S)$. The graph $G(V(G) - S)$ is denoted by $G \setminus S$. A graph G is *perfect* if $\omega(H) = \chi(H)$ for every induced subgraph H of G. A graph is *minimally imperfect* if it is not perfect but all its proper induced subgraphs are.

A *hole* is the graph induced by a chordless cycle of length at least 4. A hole is *odd* if it contains an odd number of vertices. Odd holes are not perfect since their chromatic number is 3 whereas the size of their largest clique is 2. It is easy to check that odd holes are minimally imperfect. The complement of a graph G is the graph $\overline{G}$ with the same vertex set as G, and uv is an edge of $\overline{G}$ if and only if it is not an edge of G. It is easy to check that complements of odd holes are also minimally imperfect. In the early sixties Berge [1] proposed the *Strong Perfect Graph Conjecture:* The odd holes and their complements are the only minimally imperfect graphs. This conjecture attracted much attention over the last forty years. It was proved in May 2002 by Chudnovsky, Robertson, Seymour and Thomas [9] in a very impressive paper. Claude Berge passed away in June 2002 knowing that his famous conjecture is true.

Theorem 1.1 (Strong Perfect Graph Theorem) (Chudnovsky, Robertson, Seymour and Thomas [9]) *The only minimally imperfect graphs are the odd holes and their complements.*

In this note, we survey key aspects of the proof of the Strong Perfect Graph Theorem. A *Berge graph* is a graph that does not contain an odd hole or its complement as an induced subgraph. Clearly, every perfect graph is a Berge graph. The Strong Perfect Graph Theorem states that the converse is also true: Every Berge graph is perfect. The idea of the proof is to show that every Berge graph either falls into one of four basic classes of perfect graphs, or that it has a kind of separation that cannot occur in a minimally imperfect graph.

In [1], Berge also made a weaker conjecture, which states that a graph G is perfect if and only if its complement $\overline{G}$ is perfect. This conjecture was proved by Lovász [24] in 1972. We give a short elegant proof due to Gasparyan [21].

Theorem 1.2 (Perfect Graph Theorem) (Lovász [24]) *Graph G is perfect if and only if graph $\overline{G}$ is perfect.*

Proof: Lovász [25] proved the following stronger result.

Claim 1: A graph G is perfect if and only if, for every induced subgraph H, the number of vertices of H is at most $\alpha(H)\omega(H)$.

Since $\alpha(H) = \omega(\bar{H})$ and $\omega(H) = \alpha(\bar{H})$, Claim 1 implies Theorem 1.2.

Proof of Claim 1: First assume that G is perfect. Then, for every induced subgraph H, $\omega(H) = \chi(H)$. Since the number of vertices of H is at most $\alpha(H)\chi(H)$, the inequality follows.

We give a proof of the converse due to Gasparyan [21]. Assume that G is not perfect. Let H be a minimally imperfect subgraph of G and let n be the number of vertices of H. Let $\alpha = \alpha(H)$ and $\omega = \omega(H)$. Then H satisfies

$$\omega = \chi(H \setminus v) \text{ for every vertex } v \in V(H)$$
$$\text{and } \omega = \omega(H \setminus S) \text{ for every stable set } S \subseteq V(H).$$

Let A_0 be an α-stable set of H. Fix an ω-coloring of each of the α graphs $H \setminus s$ for $s \in A_0$, let $A_1, \ldots, A_{\alpha\omega}$ be the stable sets occurring as a color class in one of these colorings and let $\mathcal{A} := \{A_0, A_1, \ldots, A_{\alpha\omega}\}$. Let $\mathbf{A}$ be the corresponding stable set versus vertex incidence

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matrix. Define $\mathcal{B} := \{B_0, B_1, \ldots, B_{\alpha\omega}\}$ where B_i is an ω-clique of $H \setminus A_i$. Let **B** be the corresponding clique versus vertex incidence matrix.

Claim 2: Every ω-clique of H intersects all but one of the stable sets in $\mathcal{A}$.

Proof of Claim 2: Let $S_1, \ldots, S_\omega$ be any ω-coloring of $H \setminus v$. Since any ω-clique C of H has at most one vertex in each S_i, C intersects all S_i's if $v \notin C$ and all but one if $v \in C$. Since C has at most one vertex in A_0, Claim 2 follows.

In particular, it follows that $\mathbf{AB}^T = J - I$ where J is the matrix filled with ones and I the identity. Since $J - I$ is nonsingular, **A** and **B** have at least as many columns as rows, that is $n \geq \alpha\omega + 1$. This completes the proof of Claim 1. □

2. Four Basic Classes of Perfect Graphs

Bipartite graphs are perfect since, for any induced subgraph H, the bipartition implies that $\chi(H) \leq 2$ and therefore $\omega(H) = \chi(H)$.

A graph L is the *line graph* of a graph G if $V(L) = E(G)$ and two vertices of L are adjacent if and only if the corresponding edges of G are adjacent.

Proposition 2.1 *Line graphs of bipartite graphs are perfect.*

Proof: If G is bipartite, $\chi'(G) = \Delta(G)$ by a theorem of König [23], where χ' denotes the edge-chromatic number and Δ the largest vertex degree.

If L is the line graph of a bipartite graph G, then $\chi(L) = \chi'(G)$ and $\omega(L) = \Delta(G)$. Therefore $\chi(L) = \omega(L)$. Since induced subgraphs of L are also line graphs of bipartite graphs, the result follows. □

Since bipartite graphs and line graphs of bipartite graphs are perfect, it follows from Lovász's perfect graph theorem (Theorem 1.2) that the complements of bipartite graphs and of line graphs of bipartite graphs are perfect. This can also be verified directly, without using the perfect graph theorem. To summarize, in this section we have introduced four classes of perfect graphs:

- bipartite graphs and their complements, and
- line graphs of bipartite graphs and their complements.

These graphs are called *basic*.

3. 2-Join, Homogeneous Pair and Skew Partition

2-Join

A graph G has a *2-join* if its vertices can be partitioned into sets V_1 and V_2, each of cardinality at least three, with nonempty disjoint subsets $A_1, B_1 \subseteq V_1$ and $A_2, B_2 \subseteq V_2$, such that all the vertices of A_1 are adjacent to all the vertices of A_2, all the vertices of B_1 are adjacent to all the vertices of B_2 and these are the only adjacencies between V_1 and V_2. 2-joins were introduced by Cornuéjols and Cunningham [17] in 1985. They gave an $O(|V(G)|^2 |E(G)|^2)$ algorithm to find whether a graph G has a 2-join.

When G contains a 2-join, we can decompose G into two blocks G_1 and G_2 defined as follows.

Definition 3.1 *If A_2 and B_2 are in different connected components of $G(V_2)$, define block G_1 to be $G(V_1 \cup \{p_1, q_1\})$, where $p_1 \in A_2$ and $q_1 \in B_2$. Otherwise, let P_1 be a shortest path from A_2 to B_2 and define* block *G_1 to be $G(V_1 \cup V(P_1))$. Block G_2 is defined similarly.*

Theorem 3.2 (2-Join Decomposition Theorem) (Cornuéjols and Cunningham [17]) *Graph G is perfect if and only if its blocks G_1 and G_2 are perfect.*

Corollary 3.3 *If a minimally imperfect graph G has a 2-join, then G is an odd hole.*

Proof: Since G is not perfect, Theorem 3.2 implies that block G_1 or G_2 is not perfect, say G_1. Since G_1 is an induced subgraph of G and G is minimally imperfect, it follows that $G = G_1$. Thus, since $|V_2| \geq 3$, V_2 induces a chordless path P_1. Therefore G is a minimally imperfect graph with a vertex of degree 2. It is well known that such a graph G is an odd hole [27]. □

Homogeneous Pair

The notion of homogeneous pair was introduced by Chvátal and Sbihi [5]. A graph G has a *homogeneous pair* if $V(G)$ can be partitioned into subsets A_1, A_2 and B, such that:

- $|A_1| + |A_2| \geq 3$ and $|B| \geq 2$.
- If a vertex of B is adjacent to a vertex of A_i then it is adjacent to all the vertices of A_i, for $i \in \{1, 2\}$.

Theorem 3.4 (Homogeneous Pair Theorem) (Chvátal and Sbihi [5]) *No minimally imperfect graph has a homogeneous pair.*

Skew Partition

A graph G has a *skew partition* if its vertices can be partitioned into four nonempty sets A, B, C, D such that there are all the possible edges between A and B and no edges from C to D. Chvátal [3] introduced skew partitions in 1985 and he conjectured that no minimally imperfect graph has a skew partition. He observed that the conjecture holds for a *star cutset*, defined to be a skew partition where $|A| = 1$.

Lemma 3.5 (Star Cutset Lemma) (Chvátal [3]) *No minimally imperfect graph has a star cutset.*

Proof: Let G_1 be the graph induced by $A \cup B \cup C$ and G_2 the graph induced by $A \cup B \cup D$. The graphs G_1 and G_2 are perfect. Let S_i be the color class of an $\omega(G)$-coloring of G_i that contains the unique node of A, for $i \in \{1, 2\}$. Then S_i meets all the $\omega(G)$-cliques of G_i, i.e. $\omega(G \setminus (S_1 \cup S_2)) < \omega(G)$. It follows that $G \setminus (S_1 \cup S_2)$ can be colored with fewer than $\omega(G)$ colors, since it is perfect. Since $S_1 \cup S_2$ is a stable set, G can be colored with $\omega(G)$ colors, a contradiction. □

Noteworthy contributions towards the skew partition conjecture were made by Hoàng [22] and Roussel and Rubio [28]. The conjecture was settled by Chudnovsky, Robertson, Seymour and Thomas [9]. They obtained it as a consequence of the Strong Perfect Graph Theorem.

Theorem 3.6 (Skew Partition Theorem) (Chudnovsky, Robertson, Seymour and Thomas [9]) *No minimally imperfect graph has a skew partition.*

In order to prove the Strong Perfect Graph Theorem, Chudnovsky, Robertson, Seymour and Thomas first proved the following weaker result.

A skew partition is *balanced* if

(i) every induced path of length at least 2 in G with ends in $A \cup B$ and interior in $C \cup D$ is even, and

(ii) every induced path of length at least 2 in $\overline{G}$ with ends in $C \cup D$ and interior in $A \cup B$ is even.

Theorem 3.7 (Chudnovsky, Robertson, Seymour and Thomas [8]) *A minimally imperfect Berge graph with smallest number of vertices cannot have a balanced skew partition.*

We give the proof of Theorem 3.7. It uses Lovász's Replication Lemma [24] which we discuss next. Incidentally, the Replication Lemma was the step that Fulkerson missed in his attempt to prove the Perfect Graph Theorem. Because Fulkerson had convinced himself that it was likely to be false, he had not tried very hard to prove it. Fulkerson [20] says: "In the Spring of 1971, I received a postcard from Berge saying that he had just heard that Lovász had a proof of the perfect graph conjecture. This immediately rekindled my interest, naturally, and so I sat down at my desk and thought again about the replication lemma. Some four or five hours later, I saw a simple proof of it."

Lemma 3.8 (Replication Lemma) (Lovász [24]) *Let G be a perfect graph and* $v \in V(G)$. *Create a new vertex* v' *and join it to v and to all the neighbors of v. Then the resulting graph G' is perfect.*

Proof: It suffices to show $\chi(G') = \omega(G')$ since, for induced subgraphs, the proof follows similarly. We distinguish two cases.

Case 1: Vertex v is contained in some maximum clique of G. Then $\omega(G') = \omega(G) + 1$. This implies $\chi(G') \leq \omega(G')$, since at most one new color is needed in G'. Clearly $\chi(G') = \omega(G')$ follows.

Case 2: Vertex v is not contained in any maximum clique of G. Consider any coloring of G with $\omega(G)$ colors and let S be the color class containing v. Then $\omega(G \setminus (S - \{v\})) = \omega(G) - 1$, since every maximum clique in G meets $S - \{v\}$. By the perfection of G, the graph $G \setminus (S - \{v\})$ can be colored with $\omega(G) - 1$ colors. Using one additional color for the vertices $(S - \{v\}) \cup \{v'\}$, we obtain a coloring of G' with $\omega(G)$ colors. □

Proof of Theorem 3.7: Let G be a minimally imperfect Berge graph with smallest number of vertices. Suppose that G has a balanced skew partition A, B, C, D. By the Star Cutset Lemma 3.5, each of A, B, C, D has cardinality at least two. Let G' be the graph obtained from G by adding a vertex v adjacent to all the vertices of A and to no other vertex of G. If G' contains an odd hole, then G has an odd path contradicting (i) in the definition of a balanced skew partition. Similarly, if $\overline{G}'$ contains an odd hole, (ii) is contradicted. Therefore G' is a Berge graph. Now consider $G_1 = G' \setminus D$ and $G_2 = G' \setminus C$. For $i \in \{1, 2\}$, the graph G_i is perfect since it is Berge and has fewer vertices than G. Replicate vertex v in G_i so that v belongs to a clique of size $\omega(G)$. By the Replication Lemma 3.8, the resulting graph R_i is perfect. Consider $\omega(G)$-colorings of R_1 and R_2 respectively. Both colorings have the same number of colors in A and assume w.l.o.g. that these colors are $1, 2, \ldots, k$. Let K be the subgraph of G induced by the vertices with colors $1, 2, \ldots, k$ and let H be the subgraph of G induced by the vertices with other colors. Since every $\omega(G)$-clique of G is in $G \setminus D$ or $G \setminus C$, the largest clique in K has size k and the largest clique in H has size $\omega(G) - k$. The graphs H and K are perfect since they are proper subgraphs of G. Color K with k colors and H with $\omega(G) - k$ colors. Now G is colored with $\omega(G)$ colors, a contradiction to the assumption that G is minimally imperfect. □

Theorem 3.7 was presented in September 2001 at a workshop in Princeton. As the next step towards Theorem 3.6, Chudnovsky and Seymour obtained the following theorem in January 2002.

Theorem 3.9 (Chudnovsky and Seymour [10]) *A minimally imperfect Berge graph with smallest number of vertices cannot have a skew partition.*

4. Decomposition of Berge Graphs

Conforti, Cornuéjols and Vušković proposed the following approach to solving the Strong Perfect Graph Conjecture.

Conjecture 4.1 (Conforti, Cornuéjols and Vušković (2001)) **(Decomposition Conjecture)** *Every Berge graph G is basic or has a skew partition, or G or* $\overline{G}$ *has a 2-join.*

Chudnovsky, Robertson, Seymour and Thomas proved the following variation of this conjecture.

Theorem 4.2 (Chudnovsky, Robertson, Seymour and Thomas [9]) **(Decomposition Theorem)** *Every Berge graph G is basic or has a skew partition or a homogeneous pair, or G or* $\overline{G}$ *has a 2-join.*

This theorem implies the Strong Perfect Graph Theorem. Indeed, suppose that the Decomposition Theorem holds and that there exists a minimally imperfect graph G distinct from an odd hole or its complement. Choose G with the smallest number of vertices. G cannot have a skew partition by Theorem 3.9. G cannot have a homogeneous pair by Theorem 3.4. Neither G nor $\overline{G}$ can have a 2-join by Corollary 3.3. Since G is a Berge graph, G must be basic by the Decomposition Theorem. Therefore G is perfect, a contradiction.

Theorem 4.2 was already known to hold in several special cases. For example, it was known when G is a Meyniel graph (Burlet and Fonlupt [2] in 1984), when G is claw-free (Chvátal and Sbihi [6] in 1988 and Maffray and Reed [26] in 1999), diamond-free (Fonlupt and Zemirline [19] in 1987), bull-free (Chvátal and Sbihi [5] in 1987), or dart-free (Chvátal, Fonlupt, Sun and Zemirline [4] in 2000). All these results involve special types of skew partitions (such as star cutsets) and, in some cases, homogeneous pairs [5]. A special case of 2-join called augmentation of a flat edge appears in [26]. In 1999, Conforti and Cornuéjols [13] used more general 2-joins to prove Conjecture 4.1 for WP-free Berge graphs, a class of graphs that contains all bipartite graphs and all line graphs of bipartite graphs. [13] was the precursor of a sequence of decomposition results involving 2-joins. The following result was obtained in February 2001.

Theorem 4.3 (Conforti, Cornuéjols and Vušković [14]) *A square-free Berge graph is bipartite, the line graph of a bipartite graph, or has a 2-join or a star cutset.*

A breakthrough occured in September 2001 when Chudnovsky, Robertson, Seymour and Thomas announced that they could prove the Decomposition Conjecture in the following important special case.

Theorem 4.4 (Chudnovsky, Robertson, Seymour and Thomas [8]) *If G is a Berge graph that contains the line graph of a bipartite subdivision of a 3-connected graph, then G has a balanced skew partition, or G or* $\overline{G}$ *has a 2-join or is the line graph of a bipartite graph.*

Given two vertex disjoint triangles a_1, a_2, a_3 and b_1, b_2, b_3, a *subdivided prism* is a graph induced by three chordless paths, $P^1 = a_1, \ldots, b_1$, $P^2 = a_2, \ldots, b_2$ and $P^3 = a_3, \ldots, b_3$, at least one of which has length greater than one, such that P^1, P^2, P^3 have no common vertices and the only adjacencies between the vertices of distinct paths are the edges of the two triangles. The next result, obtained in January 2002, is a real tour-de-force and a key step in the proof of the Strong Perfect Graph Theorem. In particular, it was needed to prove Theorem 3.9.

Theorem 4.5 (Chudnovsky and Seymour [10]) *If G is a Berge graph that contains a subdivided prism, then G is the line graph of a bipartite graph or G has a balanced skew partition or a homogeneous pair, or G or $\overline{G}$ has a 2-join.*

A *wheel* (H, v) consists of a hole H together with a vertex v, called the *center*, with at least three neighbors in H. If v has k neighbors in H, the wheel is called a *k-wheel*. A *line wheel* is a 4-wheel (H, v) that contains exactly two triangles and these two triangles have only the center v in common. A *twin wheel* is a 3-wheel containing exactly two triangles. A *universal wheel* is a wheel (H, v) where the center v is adjacent to all the vertices of H. A *triangle-free wheel* is a wheel containing no triangle. A *proper wheel* is a wheel that is not any of the above four types. These concepts were first introduced in [13]. The following theorem, obtained in May 2002, generalizes an earlier result of Zambelli presented in September 2001 and of Thomas [29].

Theorem 4.6 (Conforti, Cornuéjols and Zambelli [16]) *If G is a Berge graph that contains no proper wheel, subdivided prism or their complements, then G is basic or has a skew partition.*

The last step in proving the Strong Perfect Graph Theorem is the following diffcult theorem, also obtained in May 2002.

Theorem 4.7 (Chudnovsky and Seymour [11]) *If G is a Berge graph that contains a proper wheel, but no subdivided prism or its complement, then G has a skew partition, or G or $\overline{G}$ has a 2-join.*

Theorems 4.5, 4.6 and 4.7 imply the Decomposition Theorem 4.2, and therefore the Strong Perfect Graph Theorem. A monumental paper containing these results is now available [9].

Conforti, Cornuéjols and Vušković [15] proved a weaker version of the Decomposition Conjecture where "skew partition" is replaced by "double star cutset". A *double star* is a vertex set S that contains two adjacent vertices u, v and a subset of the vertices adjacent to u or v. Clearly, if G has a skew partition, then G has a double star cutset: Take $S = A \cup B$, $u \in A$ and $v \in B$. Although the decomposition result in [15] is weaker than Conjecture 4.1 for Berge graphs, it holds for a larger class of graphs than Berge graphs: By changing the decomposition from "skew partition" to "double star cutset", the result can be obtained for all odd-hole-free graphs instead of just Berge graphs.

Theorem 4.8 (Conforti, Cornuéjols and Vušković [15]) *If G is an odd-hole-free graph, then G is a bipartite graph or the line graph of a bipartite graph or the complement of the line graph of a bipartite graph, or G has a double star cutset or a 2-join.*

Theorem 4.8 was used by Cornuéjols, Liu and Vušković [18] to construct a polynomial time recognition algorithm for perfect graphs. Independently, Chudnovsky and Seymour [12] found a different algorithm for perfect graph recognition which does not use decomposition. Both algorithms [12], [18] build on the same companion paper [7] which performs a certain "cleaning" step in polynomial time.

A useful tool for studying Berge graphs is due to Roussel and Rubio [28]. This lemma was proved independently by Chudnovsky, Robertson, Seymour and Thomas [8], who popularized it and named it *The Wonderful Lemma*. It is used repeatedly in the proofs of Theorems 4.4-4.7.

Lemma 4.9 (The Wonderful Lemma) (Roussel and Rubio [28]) *Let G be a Berge graph and assume that $V(G)$ can be partitioned into a set S and an odd chordless path $P = u, u', \ldots, v', v$ of length at least 3 such that u, v are both adjacent to all the vertices in S and $\overline{G}(S)$ is connected. Then one of the following holds:*

(i) An odd number of edges of P have both ends adjacent to all the vertices in S.

(ii) P has length 3 and $\overline{G}(S \cup \{u', v'\})$ contains an odd chordless path between u' and v'.

(iii) P has length at least 5 and there exist two nonadjacent vertices x, x' in S such that $(V(P) \setminus \{u, v\}) \cup \{x, x'\}$ induces a path.

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ISMP2003 NEWS

ISMP2003 - The 18 international Symposium on Mathematical Programming- to be held 18 - 22 August in Copenhagen, Denmark, is now rapidly approaching.

We have received 737 abstracts for presentation, and the number of registered participants is currently 939. The organizing committee has made a serious effort to avoid no shows, and currently there are only 2 "non-paid" abstracts. Judging by the contents of the abstracts, the symposium will - as is common for this event - be of high scientific quality.

The presentations will be scheduled in 25 parallel sessions. The daily schedule starts at 9 with a block of parallel sessions followed by a plenary lecture. A block of parallel sessions initiates the afternoon followed by 3 semi-plenary lectures. Finally, on Monday August 18 and Thursday August 21, a block of parallel sessions ends the day, whereas Tuesday 19 and Wednesday 20 are "early off" days due to the Conference Dinner and the City Hall reception. The symposium ends Friday August 22 just before lunch with the last of the 17 plenary and semi-plenary lectures.

The full program as well as all other information is available on the symposium home page:www.ismp2003.dk.

We look forward to welcoming you in Copenhagen.

Jens Clausen
Chairman of the Organizing Committee

The 7th International Symposium on Generalized Convexity/Monotonicity

Hanoi, Vietnam, August 27-31, 2002.

The Symposium (GC7) was organized by the international Working Group on Generalized Convexity (WGGC) and hosted by the Hanoi Institute of Mathematics, Vietnam National Center for Natural Sciences and Technology at Hanoi, Vietnam during August 27-31, 2002. For the first time the Symposium was held outside North America and Europe reflecting the growing research activities in the Asia-Pacific region. It was sponsored by the Pacific Optimization Research Activity Group as well.

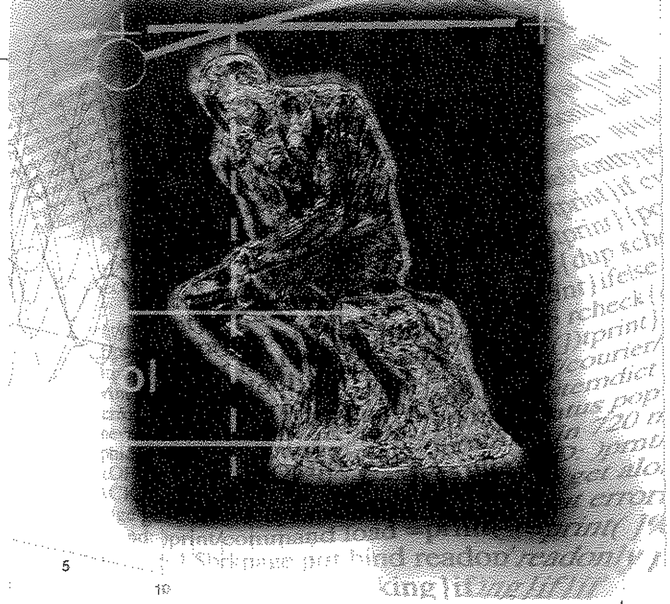
The aim of the Symposium was to provide a forum for the exchange and dissemination of new ideas in the field of generalized convexity and generalized monotonicity and their applications in optimization, control, stochastics, economics, management science, finance, engineering and other related topics. The purpose of the Symposium was fulfilled as GC7 was well represented by researchers from many parts of the world. There was a noticeable increase in the number of participants from the Asia-Pacific region. Collaboration of researchers from various countries has been an integral part of the research carried out by members of WGGC, and this was reflected by the presentations made during the conference. The sense of being a part of a large family of researchers with common interests was special. The credit of new joint works in the near future would go to the organizers of the conference and the participants who made this symposium a great success. The organizers did an excellent job of arranging a comfortable stay in Hanoi and providing the facility of using the library and internet services for all participants throughout the day.

About fifty presentations, including the invited talks, were made during the five days of the Symposium. The invited talks were presented by R.E.Burkard, Austria (combinatorial optimization), B.Mordukhovich, USA (nonsmooth analysis) and H.Tuy, Vietnam (global optimization). For program details of GC7 we refer to the web page at www.math.ac.vn/conference/gcm7/ and to the WGGC web page at www.genconv.org. Refereed Proceedings of the Symposium will appear in a volume with Kluwer Academic Publishers to be edited by A.Eberhard, N. Hadjisavvas and D.T.Luc. Information on the proceedings of the previous six symposia and on future WGGC activities is available at www.genconv.org as well.

Apart from the excellent academic atmosphere the participants had the opportunity of enjoying dinner with Vietnamese traditional music, exploring the city of Hanoi, watching the amazing puppet show at the Water Puppet Theater and tasting Vietnamese cuisine at the Banquet with a first-rate performance of Vietnamese instrumental and vocal music. Some of the participants stayed on for a two-day tour to the beautiful spot of Ha Long Bay near the sea recognized by the U.N. as a World Heritage Site. It is surrounded by many small islands of various shapes and is famous for its amazing limestone cliffs, numerous caves and unbelievable scenic beauty. This excursion was a memorable ending of the trip to Vietnam for many participants of GC7.

Sandor Komlosi, Secretary WGGC
komlosi@ktk.pte.hu

mindsharpener



We invite OPTIMA readers to submit solutions to the problems to Robert Bosch (bobb@cs.oberlin.edu). The most attractive solutions will be presented in a forthcoming issue.

Constructing Nontransitive Dice

Robert A. Bosch

April 3, 2003

Two gamblers have decided to use the (unfolded) dice displayed in Figure 1 to settle an argument. They've decided that

- they'll each pick one die and roll it once,
- gambler 1 will pick first, and
- whoever rolls the higher number will win.

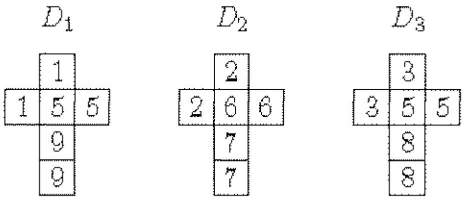


Figure 1

These dice are *nontransitive*. If gambler 1 picks D_1, gambler 2 should pick D_2. If gambler 1 picks D_2, gambler 2 should pick D_3. And if gambler 1 picks D_3, gambler 2 should pick D_1. No matter which die gambler 1 picks, gambler 2 will win with probability 5/9 (since $\text{Prob}(D_1 \succ D_3) = \text{Prob}(D_2 \succ D_1) = \text{Prob}(D_3 \succ D_2) = 5/9$).

The well-known statistician Bradley Efron was the first to design sets of nontransitive dice, and Martin Gardner was the first to popularize them (see chapter 22 of [1]).

Problems

Interested readers may enjoy trying to solve the following problems:

1. Devise an integer programming formulation or a constraint programming formulation for constructing nontransitive dice.
2. Use the formulation to find a set of three nontransitive dice that has the following properties: (i) each face has a number between 1 and 18 on it, (ii) each number in this range appears on exactly one face, and (iii) $\text{Prob}(D_1 \succ D_3) \le \text{Prob}(D_2 \succ D_1) \le \text{Prob}(D_3 \succ D_2)$. Maximize $\text{Prob}(D_1 \succ D_3)$.

Slither Link Revisited

The previous *Mindsharpener* was concerned with *slither link* puzzles. In a slither link puzzle, the goal is to find a cycle that consists of horizontal and vertical line segments and satisfies the puzzle's adjacency conditions: for each square s and for every number a, if square s has the number a in it, then s must be adjacent to precisely a segments of the cycle. See Figure 2 for an example; see Figure 3 for its solution.

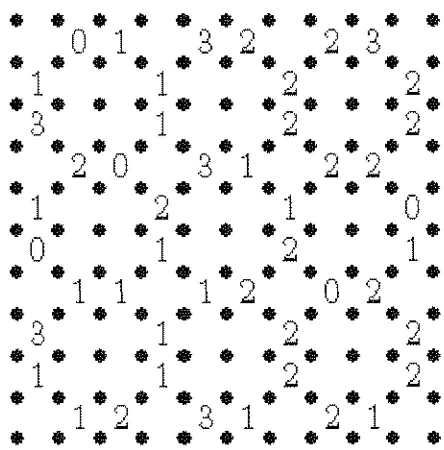


Figure 2

It is easy to formulate an IP that can be used to solve slither link puzzles. In an n x n puzzle, there are n rows and n columns of squares and $n + 1$ rows and $n + 1$ columns of points. We number the rows and columns of squares from 1 to n and the rows and columns of points from 0 to n. For each $0 \le i, j \le n$, we let $p_{i,j}$ equal 1 if the cycle visits point (i, j) and 0 if not. For each $0 \le i \le n$

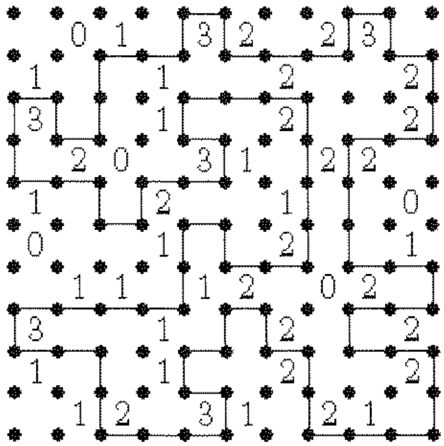


Figure 3

and $0 \le j \le n-1$, we let $h_{i,j}$ equal 1 if the horizontal line segment connecting points (i, j) and $(i, j+1)$ is a part of the cycle and 0 if not. For each $0 \le i \le n-1$ and $0 \le j \le n$, we let $v_{i,j}$ equal 1 if the vertical line segment connecting points (i, j) and $(i+1, j)$ is a part of the cycle and 0 if not.

For each point (i, j), we need a "degree" constraint. If (i, j) is an interior point (i.e., $0 < i, j < n$) the degree constraint is

$$h_{i,j-1} + h_{i,j} + v_{i-1,j} + v_{i,j} = 2p_{i,j}.$$

This constraint says two things: (1) if the cycle visits point (i, j), then exactly two line segments of the cycle are incident to (i, j), and (2) if the cycle doesn't visit point (i, j), then none of the line segments of the cycle are incident to (i, j). The degree constraints for edge points and corner points are similar.

For each square that contains a number, we need another constraint. If square (i, j) contains the number a, then we need to include the constraint

$$h_{i-1,j-1} + h_{i,j-1} + v_{i-1,j-1} + v_{i-1,j} = a.$$

This constraint ensures that exactly a segments of the cycle are adjacent to square (i, j).

Our solution strategy is to minimize the length of the "tour" subject to all of these constraints. If we find that there are subtours, we simply add constraints to eliminate them. (None were needed to produce the solution displayed in Figure 3.)

References

[1] M. Gardner, *The Colossal Book of Mathematics*, WW Norton, 2001.

Announcement

The Fourth International Conference on "Frontiers in Global Optimization" (organized by C. Floudas and P. Pardalos) took place June 8-12, 2003 in Santorini, Greece. About 85 active researchers from all over the world participated in the conference. The conference focused on deterministic methods for global optimization, stochastic methods for global optimization, distributed computing methods in global optimization, and applications of global optimization in all branches of applied science and engineering, computer science, computational chemistry, structural biology, and bioinformatics. A refereed conference book with selected papers based on talks presented at the conference will be published by Kluwer Academic Publishers later this year.

Conference

Multiscale Optimization Methods and Applications

February 26-28, 2004

The Center for Applied Optimization at the University of Florida, in conjunction with the 2003/2004 Special Year Mathematics Program, is hosting a conference entitled "*Multiscale Optimization Methods and Applications,*" February 26-28, 2004. For information about the conference, please see the web site: http://www.math.ufl.edu/special03// or contact one of the organizers:

Timothy Davis (davis@cise.ufl.edu)
William Hager (hager@math.ufl.edu)
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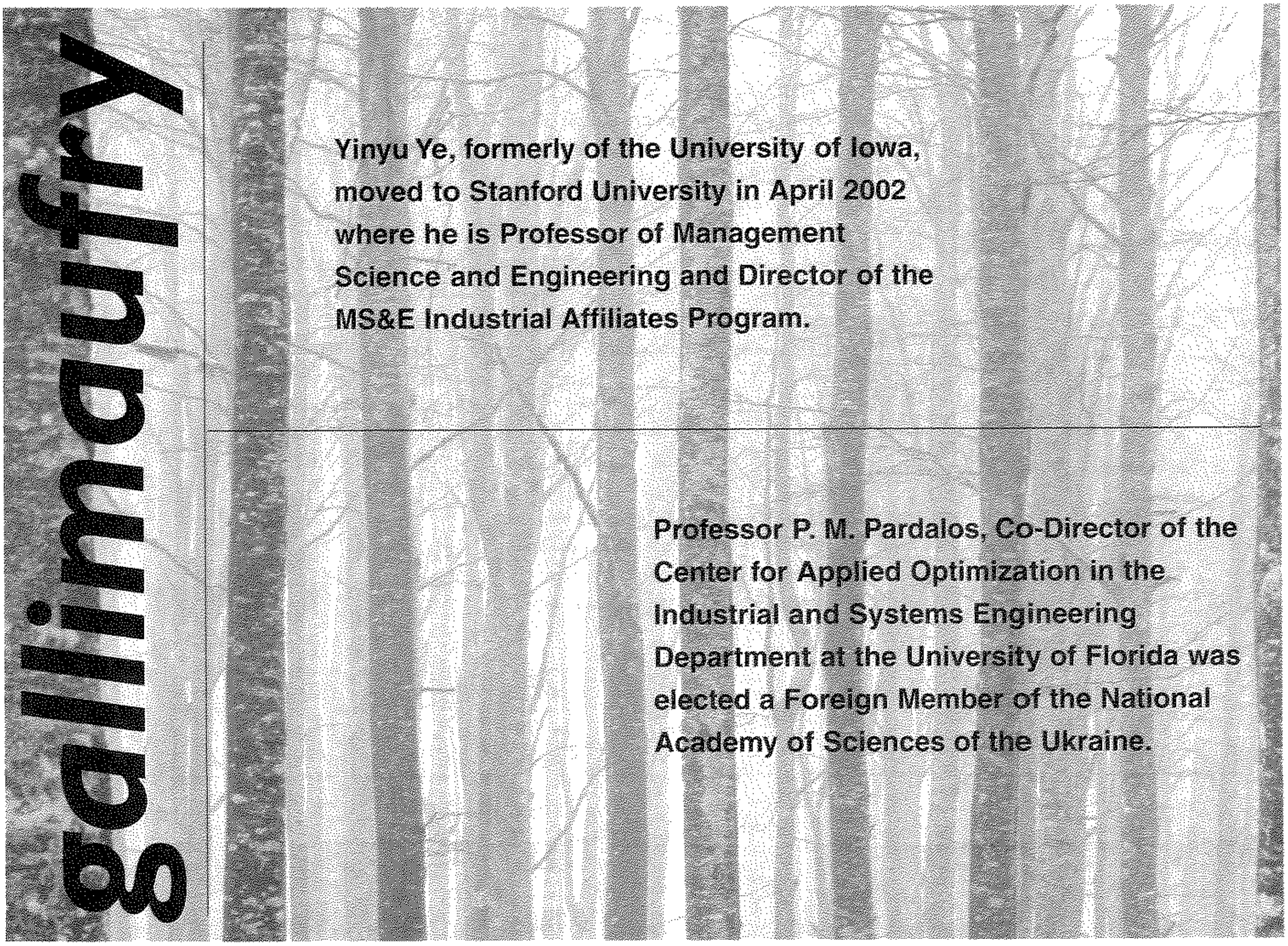

Yinyu Ye, formerly of the University of Iowa, moved to Stanford University in April 2002 where he is Professor of Management Science and Engineering and Director of the MS&E Industrial Affiliates Program.

Professor P. M. Pardalos, Co-Director of the Center for Applied Optimization in the Industrial and Systems Engineering Department at the University of Florida was elected a Foreign Member of the National Academy of Sciences of the Ukraine.

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