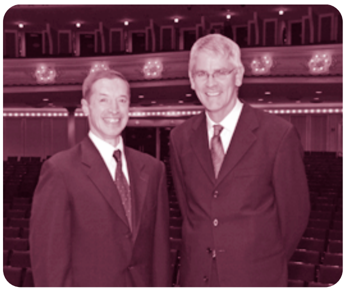
OPTIMA 80

Mathematical Programming Society Newsletter

Steve Wright

MPS Chair's Column

September 6, 2009. I'm writing in the wake of ISMP 2009 in Chicago, which ended last weekend. All involved in the organization were delighted with how well the meeting ran, and hope that all attendees (over 1200 of them) found it to be a rich and valuable experience, both professionally and personally. There are too many highlights to recount here; suffice to say the event's success indicates that our field of optimization is in excellent health. I have no doubt that our research will continue to increase in depth and breadth in the coming years – and in its influence on research and understanding in many other areas. I thank profoundly the many people who worked so hard to make the event a success, starting with John Birge, who chaired the organization and program committees.



John Birge, ISMP 2009 Chair (left) and Steve Wright, MPS Chair, at the Open Ceremony of ISMP 2009, August 23, 2009, Orchestra Hall, Chicago. Photo by Chris Buzanis, CGPA Photography.

Many members would have noted that this ISMP followed a different framework from past meetings, which were organized at campus locations by independent local committees. The unsuitability of campus sites in Chicago pushed us toward the downtown location and toward a more professional mode of organization involving our fellow society INFORMS, which was able to sign contracts when MPS was unable to do so. We found INFORMS to be a great organizational partner. The benefits of high quality meeting rooms, top-class social and ceremonial events, and the vibrant downtown location that we enjoyed during this symposium speak for themselves. Its success has more than justified the decision to try an alternative model for ISMP organization this time around. This model provides an additional option for those who are thinking about organizing future symposia.

One novel feature of this symposium was the daily newsletter Optima@ISMP, which was published on each day of the meeting. I thank editor/publisher Leah Lavelle for her great work on the newsletter, which was popular and practical, with more than a touch of class.

The ISMP web site will stay up indefinitely. Its URL may change, but it will be accessible through the society's web site mathprog.org. The five issues of Optima@ISMP have been posted there, and some photos will appear in the near future.

A major news item from ISMP was the announcement about the 2012 symposium, which will be held at TU Berlin. I thank Martin Skutella and his colleagues at other Berlin institutions for taking on this responsibility, and have no doubt that we'll be celebrating another great symposium three years from now. I'm grateful too to the groups at London and Istanbul, who also prepared excellent bids.

Following MPS tradition, we held our major council and business meetings of the three-year cycle during the symposium. I welcome the chair-elect Philippe Toint and treasurer-elect Juan Meza, who will take their positions in August 2010. (Philippe serves as vice-chair until that date.) The new councilors-at-large Jeff Linderoth, Claudia Sagastizábal, Martin Skutella, and Luis Vicente took office during ISMP. The society will be in excellent hands during the new 3-year term.

The new constitution was ratified at the business meeting, a step in our efforts to overhaul the society's legal framework. The by-laws too have undergone major changes during the past year, to bring them into line with governmental expectations for professional nonprofit organizations, to reflect current practices, and to include new material on such matters as ICCOPT and the mathprog.org web site. Besides their legal function, the by-laws serve as a reference guide for future leaders of the society on how we carry out our most important functions, especially publications, prizes, and meetings.

During ISMP, the Committee on Stochastic Programming (COSP) formally became a technical section of MPS. COSP was founded originally as a standing committee of MPS, but the by-laws of the two organizations fell out of sync over the years, a situation that has now been remedied by this new status. We look forward to working on projects of mutual benefit with the members of this very active group, which includes many of our most distinguished members and which represents a research area whose importance continues to surge.

Our journals *Mathematical Programming, Series A and B*, recently reported impact factors and article influence scores that are among the leaders in the Applied Mathematics class. Kudos to the editorial staff for their great work in maintaining the quality and reputation of our publications. *Mathematical Programming Computation*, meanwhile, is off to a great start, with its very first published paper (by Tobias Achterberg) receiving this year's Beale-Orchard-Hays Prize during the ISMP opening ceremony. At the time of writing, our publications are featured heavily on Springer's mathematics home page.

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After sporadic discussion over many years, a consensus emerged clearly at the council and business meetings that the time had come to change our society's name. Our current name has become increasingly old-fashioned and difficult to explain. While the alternative term "optimization" is not fully inclusive of our activities and is also ambiguous in certain quarters, it has much more universal recognition as a title for our field of research. Possibly the most obvious new name for the society is "Mathematical Optimization Society." The feeling is that we should retain the current names for our journals, to avoid archival confusion, but that a subtitle could be added to indicate that they are journals of our society. Concrete proposals will be discussed in the months ahead. As with all other issues concerning the society, you are welcome to make your thoughts known to me or to other members of the society's leadership.

Finally, I mention that renewal notices for 2010 membership will be sent in October. I urge you to renew your membership and continue your participation in the society during this time of vitality and growth.

Santanu S. Dey and Andrea Tramontani

Recent Developments in Multi-Row Cuts

1 Introduction

A classical way to strengthen linear programming relaxations of mixed integer linear programs (MIP) is to add linear inequalities known as cutting planes or cuts. Over the years, cutting planes have proven to be indispensable tools in solving MIPs. One approach for generating cutting planes for a given MIP is to use the facet-defining inequalities of the convex hull of its feasible solutions. However, since in the case of general MIPs little structure can be assumed, it is difficult, if not impossible, to analyze the polyhedral properties of the convex hull of the feasible solutions. Therefore, the feasible region is typically relaxed so as to obtain a set that is more amenable to analysis. One common relaxation often considered in the literature is the 'single constraint relaxation'. In this scheme the cut generation procedure may be viewed as a two step process. In the first step, all but one of the constraints of the MIP are dropped. Alternatively, various constraints are multiplied by suitable weights and then added to obtain a single implied constraint. Then a black box for generating a cutting plane from a single constraint (often using information on individual variable restrictions such as nonnegativity, bounds, and integrality) is invoked to derive a cut. As the single constraint defines a relaxation of the original MIP, the resulting cut is valid for the original MIP. Gomory mixed integer cuts (GMI) (Gomory [26]), mixed integer rounding inequalities (MIR) (Nemhauser and Wolsey [34]), and cover cuts (Balas [7], Wolsey [36]) are some examples of cuts generated using this paradigm.

Very recently, a series of papers have focussed on the possibility of generating cuts using more than one row of the simplex tableau (or constraints) simultaneously. Several interesting theoretical results have been presented in this direction, often revisiting and recalling other important results discovered more than 40 years ago.

The paradigm for generating cutting planes from multiple rows of simplex tableau remains similar to that of generating cuts based on one row. Typically, we start with a simplex tableau of the form

$$\begin{aligned} x_B &= x_B^* + \sum_{j\in N} r^j x_j, \\ x &\geq 0, \\ x_j &\in \mathbb{Z},\ j \in J, \end{aligned} \tag{1}$$

where B (resp. N) denotes the set of basic (resp. nonbasic) variables and the current incumbent solution $(x_B, x_N) := (x_B^*, 0)$ is assumed to be integer infeasible. As before, a relaxation of (1) is generated. However, now this new relaxation may contain more than one constraint. Then, facet-defining inequalities (or the closely related *extreme inequalities*) for this relaxation are generated. These inequalities are valid for the original MIP by virtue of the fact that they are valid for a relaxation.

The art lies in obtaining a relaxation that is both easy to analyze and yet strong enough to generate potent cutting planes. In this paper we review some of the relaxations (and their analysis) that are closely related to the group relaxation, originally invented by Gomory [28]. In order to construct the group relaxation, the nonnegativity requirement on the basic variables x_B is relaxed and new nonnegative variables are introduced in (1). In Section 2, we review the group relaxation, define some generic concepts used in the rest of the paper, and summarize some recent approaches for generating cutting planes based on multiple constraints using the classical group relaxation.

Andersen et al. [3] and Borozan and Cornuéjols [12] considered further relaxing the group relaxation by removing the integrality restriction on the nonbasic variables. This relaxation lead to a significantly different perspective on the multi-row cutting planes. In particular, a wonderful connection between extreme inequalities of these relaxations and *lattice-free convex sets* has been established via the principle of *Intersection cuts* (Balas [6]). Results related to various variants of the relaxation introduced in [3] and [12] form the bulk of this paper and are discussed in Section 3. In Section 4, we review some results evaluating the strength of these new classes of inequalities and comparing their properties with respect to *split inequalities* (Cook et al. [14]). Many questions remain open and some of them are highlighted in Section 5.

We finally note that there are numerous approaches to generating cuts based on multiple rows. This review if confined only to the recent approaches of cut generation based on the group relaxation.

2 The Group Relaxation

For the sake of simplicity we assume that the set (1) is nonempty. The first step in the construction of the Group relaxation is the construction of the *Corner relaxation*. This relaxation is obtained by dropping the nonnegative restrictions on all the basic variables and considering a subset of m rows of (1) associated with basic integer-constrained variables (i.e., a subset of variables x_i with $i \in B \cap J$), thus obtaining

$$\begin{aligned} & x_B = x_B^* + \sum_{j\in N\cap J} r^j x_j + \sum_{j\in N\setminus J} r^j y_j, \\ & x_B \in \mathbb{Z}^m, \\ & x_j \geq 0, \quad x_j \in \mathbb{Z}, j \in N\cap J, \\ & y_j \geq 0, \qquad\qquad j \in N \setminus J, \end{aligned} \tag{2}$$

where now B denotes the set of m basic integer-constrained variables corresponding to the selected rows. (Hence forth continuous variables are represented using the letter y to distinguish them from integer variables represented using x.) Note that if the simplex tableau is non-degenerate, then the set of constraints that are not active is exactly the set of nonnegativity constraints on the basic variables. Thus, the motivation for constructing the Corner relaxation is that the constraints that are currently active at the solution $(x_B, (x, y)_N) := (x_B^*, (0, 0))$ are possibly more important, while dropping the nonactive constraints may simplify the analysis of the resulting set (2). Indeed, Gomory [27] presented necessary conditions for the optimal solution of the corner relaxation to be the same as the optimal solution of the original integer program. This is known as the Asymptotic Theorem.

It is customary to replace x_B^* by f, where $f_i = (x_B^*)_i - \lfloor (x_B^*)_i \rfloor$, as this amounts to just translating x_B by an integral vector. Moreover, since x_B is a free integer vector, it can be verified that if $r_{j_1} - r_{j_2} \in \mathbb{Z}^m$, then in any valid strong inequality for (2) the two variables x_{j_1} and x_{j_2} $(j_1, j_2 \in J)$ will have the same coefficient. Therefore, we can replace the column r^j corresponding to an integer vector x_j $(j \in J)$ by a vector of its fractional components.

The *group relaxation* (also called the *master group relaxation*) is obtained by the addition of more variables to the corner relaxation, i.e. we consider the set

$$\begin{aligned} x_B &= f + \sum_{r^j \in G} r^j x_j + \sum_{r^j \in W} r^j y_j, \\ x_B &\in \mathbb{Z}^m, \\ x_j &\in \mathbb{Z},\ j \in G, \\ x_G, y_W &\geq 0 \text{ and they have finite support,} \end{aligned} \tag{3}$$

where $\{r^j | j \in J \cap N\} \subseteq G$ and $\{r^j | j \in N \setminus J\} \subseteq W$. Since any feasible solution of (2) can be used to construct a solution of (3) by setting the new variables to zero, the projection of (3) onto the space of the $(x_B, (x,y)_N)$ variables is a relaxation of (2).

Before proceeding we present some notation. We refer to the set of vectors $[0\ 1)^m$ as I^m. We use the symbols $\oplus$ and $\ominus$ to denote addition and substraction modulo 1 componentwise respectively. Given a vector $v \in \mathbb{R}^m$, we let $\mathcal{F}(v)$ be a vector belonging to I^m where the i^{th} component of $\mathcal{F}(v)$ is $v_i (\text{mod } 1)$.

Gomory [28] proposed to use the columns of the integer variables from a set G in (3) where $G \subseteq I^m$, $\{r^j | j \in J \cap N\} \subseteq G$, and the elements of G are closed under the $\oplus$ operations. Observe that (3) has a new condition that x_G and x_W should have finite support, i.e. only a finite number of components of x_G and x_W are permitted to be positive in any feasible solution. This condition is added to avoid technical difficulties in the case when G or W are not finite sets. The set of feasible solutions of (3) is denoted as $R(f, G, W)$ in this review.

Definition 2.1. *Since all the x_B variables can be written in terms of the x_G and y_W variables using the first equation in (3), it is customary to write the valid inequality in terms of the x_G and y_W variables. Hence, a valid inequality for $R(f, G, W)$ is defined as a pair of functions $\phi : G \to \mathbb{R}_+$ and $\pi : W \to \mathbb{R}_+$ such that $\sum_{r^j \in G} \phi(r^j) x_j + \sum_{r^j \in W} \pi(r^j) y_j \geq 1$ is valid for all $(x_B, x_G, y_W) \in R(f, G, W)$. Here $\phi(r^j)$ represents the coefficient of the variable x_j, which in turn is the variable corresponding to the column r^j in $R(f, G, W)$. A valid inequality is called a* minimal inequality *if it is not dominated by any other inequality, i.e. (ϕ, π) is a minimal inequality if there does not exist a valid inequality (ϕ', π') such that $(\phi', \pi') \neq (\phi, \pi)$ and $\phi'(u) \leq \phi(u) \forall u \in G$ and $\pi'(w) \leq \pi(w) \forall w \in W$. A valid inequality is called an* extreme inequality *if it cannot be written as a convex combination of two distinct valid inequalities.*

Often valid inequalities are also referred to as valid functions. These definitions carry through to almost all the models/relaxations reviewed in paper. Note also that extreme inequalities and facet-defining inequalities are equivalent concepts when the group G is finite. Indeed, Gomory and Johnson [29, 30] prove the following fundamental result: Every extreme inequality is a minimal inequality.

The advantage of constructing the group relaxation is twofold: The polyhedral analysis of the corner relaxation presented in (2) is 'messy' since it depends on the data r^j, $j \in N$. On the other hand, the analysis of (3) is clean and elegant, since it contains all possible interesting columns and is effectively 'data independent'. The second advantage is a little more subtle and it is due to the following result by Gomory [28]: All the facet-defining inequalities of (2) can be extracted from the *extreme inequalities* of (3).

We now present one representative result about minimal inequalities of the master group relaxation that illustrate the 'niceness' of these structures.

Theorem 2.1 ([31]). *A valid inequality (ϕ, π) is a minimal inequality for $R(f, I^m, \mathbb{R}^m)$ if and only if $\phi : I^m \to \mathbb{R}_+$ and $\pi : \mathbb{R}^m \to \mathbb{R}_+$ satisfy the following conditions: $i)$ $\phi(u) + \phi(v) \geq \phi(u \oplus v)$ $\forall u, v \in I^m$, $ii)$ $\phi(u) + \phi(\mathcal{F}(-f) \ominus u) = 1$ $\forall u \in I^m$, and $iii)$ $\pi(w) = \lim_{h \downarrow 0} \frac{\phi(\mathcal{F}(hw))}{h}$ $\forall w \in \mathbb{R}^m$.*

A number of families of group cuts have been discovered and proven to be extreme based on the bedrock of Theorem 2.1 and its variants. While many of these results are primarily for the one-row group relaxation (i.e. the case where $m = 1$), recently some families of multi-row cutting planes have been proven to be extreme; see Dey and Richard [18, 19]. The results in [18, 19] present two families of cutting planes that use one-row or multi-row cuts as their building block to construct extreme inequalities for the m-row group relaxation. The first family corresponds to aggregation of rows. The second family corresponds to a more intricate sequence of cut generation from single and multiple rows applied in specific sequence which ultimately leads to extreme inequalities for m-row group relaxations. We refer the reader to [18, 19] for the details.

3 Corner and Group Relaxations together with 'Continuous Nonbasic Variables' Relaxation

3.1 Intersection Cuts and Lattice-free Convex Sets

Before presenting the results about different relaxations that are closely related to the group relaxation, we discuss a generic principle of generating cutting planes known as *intersection cutting planes*, invented by Balas [6]. We illustrate the principle of constructing intersection cuts for pure integer programs first: Suppose that we are given a convex relaxation P of S, where S is the set of feasible solutions of a pure integer program. Let $f \in P \setminus \text{conv}(S)$ be a point which we would like to separate. Let M be a convex set containing f, such that no feasible solution of S lies in the interior of this set, i.e., $S \cap \text{int}(M) = \emptyset$. Then the relaxation P can be strengthened by computing $\text{conv}(P \setminus \text{int}(M))$. The resulting inequality obtained by this operation separates f and is called an intersection cut. Thus, convex sets which do not contain integer points in their interior can be used to generate intersection cuts for integer programs (and also for MIPs as discussed in the next section). We next define these sets formally.

Definition 3.2 ([33]). *A set $M \subseteq \mathbb{R}^m$ is called lattice-free if $\text{int}(M) \cap \mathbb{Z}^m = \emptyset$. A lattice-free convex set M is maximal if there exists no lattice-free convex set $M' \neq M$ such that $M \subsetneq M'$.*

Maximal lattice-free convex sets are very structured sets. The following characterization is due to Lovász [33] and Basu et al. [10].

Theorem 3.2 ([33], [10]). *Let M be a maximal lattice-free convex set in $\mathbb{R}^m$. Then M is either an irrational affine hyperplane in $\mathbb{R}^m$ or a full-dimensional lattice-free polyhedron of the form $M = P + L$, where P is a polytope, L is a rational linear space, $\dim(M) = \dim(P) + \dim(L)$. In the second case, M has at most 2^m facets and there is an integral point in the relative interior of each facet of M.*

In two dimensions (i.e., when $m = 2$), full-dimensional maximal lattice-free convex sets can be classified as follows. (See Figure 1.)

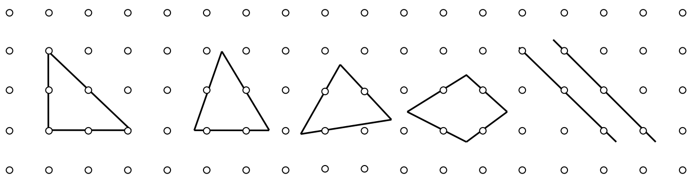

Figure 1. *Maximal lattice free sets*

Proposition 3.1. *Let M be a full-dimensional maximal lattice-free convex set in $\mathbb{R}^2$. Then M is one of the following:*

1. *A split set $\{(x_1, x_2) \mid b \le a_1x_1 + a_2x_2 \le b+1\}$ where a_1 and a_2 are coprime integers and b is an integer,*
2. *A triangle with a least one integral point in the relative interior of each of its edges, which in turn is either:*
   (a) *A type 1 triangle: triangle with integral vertices and exactly one integral point in the relative interior of each edge,*
   (b) *A type 2 triangle: triangle with at least one fractional vertex v, exactly one integral point in the relative interior of the two edges incident to v and at least two integral points on the third edge,*
   (c) *A type 3 triangle: triangle with exactly three integral points on the boundary, one in the relative interior of each edge.*
3. *A quadrilateral containing exactly one integral point in the relative interior of each of its edges.*

3.2 Two-Row Corner Relaxation with Continuous Nonbasic Variables

Andersen et al. [3] considered relaxing the integrality of all the nonbasic variables in the two-row corner relaxation, i.e. they considered the set $R(f, \emptyset, W)$:

$$x_B = f + \sum_{r^j \in W} r^j y_j, \quad x_B \in \mathbb{Z}^2, \quad y_j \ge 0 \ \forall j, \tag{4}$$

where $f \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$, $|W| = k$ is finite, and $r^j \in \mathbb{Q}^2 \setminus \{(0,0)\}$ for all $r^j \in W$. One motivation for this relaxation is the following: successful one-row cuts like MIR can be explained by considering the effect of integrality of one free integer variable (see derivation in Wolsey [37]) in a simple one-row system. By removing all the integrality requirement on the nonbasic variables, we are left with two free integer basic variables. This model is thus expected to capture the effect produced by two rows and two integer variables.

Non-trivial valid inequalities for $R(f, \emptyset, W)$ can be generated using maximal lattice-free convex sets containing f via intersection cuts: Let $M \subseteq \mathbb{R}^2$ be a maximal lattice-free convex set containing f in its interior. Consider the set $M' = \{(u, v) \in \mathbb{R}^2 \times \mathbb{R}^{|W|} \mid u \in M\}$. Let R^0 be the continuous relaxation of (4). Then we can generate a cut for (4) by computing $\mathrm{conv}(R^0 \setminus \mathrm{int}(M'))$. It can be verified that this is equivalent to generating the cut $\sum_{r^j \in W} \pi(r^j) y_j \ge 1$ where the cut coefficients $\pi(r^j)$ are computed as

$$\pi(r^j) = \begin{cases} \lambda, & \text{if } \exists \lambda > 0 \text{ s.t. } f + \frac{1}{\lambda} r^j \in \text{ boundary } (M) \\ 0, & \text{if } r^j \text{ belongs to the recession cone of } M. \end{cases} \tag{5}$$

We next show an interesting example of the set (4) and a cut of the form (5).

Example 3.1 ([14], [3]). *Consider the simple MIP*

$$\begin{array}{lll} & \max & t \\ \text{s.t.} & (c_1) & t \le x_1, \\ & (c_2) & t \le x_2, \\ & (c_3) & t + x_1 + x_2 \le 2, \\ & & x \in \mathbb{Z}^2 \text{ and } t \in \mathbb{R}^1_+. \end{array} \tag{6}$$

By introducing nonnegative slack variables s_1, s_2 and s_3 in constraints (c_1), (c_2), and (c_3) respectively, the simplex tableau corresponding to the optimal vertex $x^* = (\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$ of the LP relaxation of (6) reads

$$\begin{array}{rcllll} t & = & \frac{2}{3} & -\frac{1}{3}s_1 & -\frac{1}{3}s_2 & -\frac{1}{3}s_3 \\ x_1 & = & \frac{2}{3} & +\frac{2}{3}s_1 & -\frac{1}{3}s_2 & -\frac{1}{3}s_3 \\ x_2 & = & \frac{2}{3} & -\frac{1}{3}s_1 & +\frac{2}{3}s_2 & -\frac{1}{3}s_3 \\ & & \multicolumn{4}{l}{x \in \mathbb{Z}^2, t \in \mathbb{R}^1_+, s \in \mathbb{R}^3_+.} \end{array} \tag{7}$$

Cook et al. [14] have shown that a cutting plane algorithm based on split cuts does not suffice to generate the cut needed to solve the toy problem (6) in a finite number of iterations. If however the last two rows of (7) are considered simultaneously (i.e., a relaxation of (7)), then the cut required to solve the problem, namely

$$\frac{1}{2}s_1 + \frac{1}{2}s_2 + \frac{1}{2}s_3 \ge 1 \text{ or equivalently } t \le 0, \tag{8}$$

can be immediately derived as an intersection cut. Andersen et al. [3] show that the cut (8) is in fact an intersection cut arising from the lattice-free triangle with vertices $v_1 = (0,0)$, $v_2 = (2,0)$ and $v_3 = (0,2)$ using (5).

Now we present a result modified from [3]. This result presents necessary conditions for inequalities to be facet-defining for (4).

Theorem 3.3 ([3]). *Let $\sum_{r^j \in W} \pi(r^j) y_j \ge 1$ be a facet-defining inequality for (4). Then $\pi(r^j) \ge 0 \quad \forall j$. Let*

$$\begin{array}{lll} L_\pi = \{ & u \in \mathbb{R}^2 \mid \exists y \in \mathbb{R}^{|W|}_+, \text{s.t.} & \\ & u = f + \sum_{r^j \in W} r^j y_j, \sum_{r^j \in W} \pi(r^j) y_j \le 1 & \}. \end{array} \tag{9}$$

If $\pi(r^j) = 0$ for some j, then L_π is the subset of a split set and the inequality is a split inequality. If $\pi(r^j) > 0$ for all j, then L_π is a lattice-free triangle or a lattice-free quadrilateral.

Constructing the inequality $\sum_{r^j \in W} \pi(r^j) y_j \ge 1$ by using a maximal lattice-free convex set M and (5), and then computing L_π using (9), we typically obtain that $L_\pi \subseteq M$ where L_π may not be a maximal lattice-free convex set. Thus, by presenting a set of shapes of L_π when π is facet-defining, Theorem 3.3 gives necessary conditions for facet-defining inequalities for $R(f, \emptyset, W)$.

Cornuéjols and Margot [15] present sufficient conditions for facet-defining inequalities for $R(f, \emptyset, W)$. In order to present these conditions we require some definitions. Let M be a maximal lattice-free triangle or quadrilateral with f in its interior. For $i \in \{1, \ldots, |W|\}$, let p^i be the intersection of the ray $f + \lambda r^i$, $\lambda \ge 0$ with the boundary of M, i.e., $p^i = f + \frac{r^i}{\pi(r^i)}$. The point p^i is called the *boundary point* for r^i. A boundary point p^i is called *active* if it is integral or if there exists another boundary point p^j such that a strict convex combination of p^i and p^j is an integral point. Given a set of boundary points T, a point $p^i \in T$ is called *uniquely active* if there exists an unique point $p^j \in T$ such that a strict convex combination of p^i and p^j is an integral point. The following *Reduction Algorithm* from Cornuéjols and Margot [15] helps in simplifying the characterization of facets of $R(f, \emptyset, W)$: $i)$ Let $T := \{p^1, \ldots, p^k\}$. $ii)$ While there exists $p \in T$ such that p is active and p is a convex combination of other points in T, remove p from T. $iii)$ While there exists a uniquely active $p \in T$, remove p from T. $iv)$ If $T = \{p, q\}$ and the segment pq contains at least two integral points, remove both p and q from T.

The *ray condition* is said to holds if $T = \emptyset$ at termination of the Reduction Algorithm.

Theorem 3.4 ([15]). *Let M be a maximal lattice-free convex set containing f in its interior. Then the inequality $\sum_{r^i \in W} \pi(r^i) y_i \geq 1$ generated using (5) is extreme if:*

1. *M is a split set, where the recession direction of M is r^i for some $i \in \{1, \ldots, |W|\}$,*
2. *M is a maximal lattice-free triangle and*
   (a) *there exist r^{i_1}, r^{i_2}, r^{i_3} such that the points $f + \frac{r^{i_j}}{\pi(r^{i_j})}$, $j \in \{1, 2, 3\}$ are the three corner points of M or*
   (b) *the ray condition holds,*
3. *M is a maximal lattice-free quadrilateral and there exist r^{i_1}, $r^{i_2}, r^{i_3}, r^{i_4}$ such that the points $p^j := f + \frac{r^{i_j}}{\pi(r^{i_j})}$, $j \in \{1, 2, 3, 4\}$ are the four corner points of M, and there does not exist an $h \in \mathbb{R}_+$ such that*

$$\frac{|b^j - p^j|}{|b^j - p^{j+1}|} = \begin{cases} h & \text{if } j = 1, 3 \\ \frac{1}{h} & \text{if } j = 2, 4, \end{cases} \tag{10}$$

where b^j is the integer point lying on the line segment $p^j p^{j+1}$.

Thus, Theorem 3.3 and Theorem 3.4 give the complete characterization of extreme equalities for (4) using the properties of lattice-free convex sets in $\mathbb{R}^2$.

3.3 Multi-Row Master Infinite Group Relaxation with Continuous Nonbasic Variables

Borozan and Cornuéjols [12] considered relaxing the integrality of all the nonbasic variables in the m-row semi-infinite master group relaxation, i.e. they considered the set $R(f, \emptyset, \mathbb{Q}^m)$:

$$\begin{aligned} &x_B = f + \textstyle\sum_{r^j \in \mathbb{Q}^m} r^j y_j, \\ &x_B \in \mathbb{Z}^m, \quad y \geq 0 \text{ and has a finite support,} \end{aligned} \tag{11}$$

where $f \in \mathbb{Q}^m \setminus \mathbb{Z}^m$.

Theorem 3.5 ([12]). *Any valid inequality for $R(f, \emptyset, \mathbb{Q}^m)$ can be written in the form $\sum_{r^j \in \mathbb{Q}^m} \pi(r^j) y_j \geq 1$, where $\pi : \mathbb{Q}^m \longrightarrow \mathbb{Q}_+ \cup \{+\infty\}$. A minimal valid function π for $R(f, \emptyset, \mathbb{Q}^m)$ is nonnegative, piecewise linear, positively homogeneous and convex. Furthermore the set $cl(L_\pi) = cl\{u \in \mathbb{Q}^m \,|\, \pi(u - f) \leq 1\}$ is a full-dimensional maximal lattice-free convex set containing f. Conversely, for any full-dimensional maximal lattice-free convex set $M \subset \mathbb{R}^q$ containing f, there exists a minimal valid function π for $R(f, \emptyset, \mathbb{Q}^m)$ such that $cl(L_\pi) = M$. When f is in the interior of M, this function is unique and can be computed using (5).*

Theorem 3.5 again illustrates the relationship between minimal inequalities and lattice-free convex sets. Any minimal valid inequality for $R(f, \emptyset, \mathbb{Q}^m)$ arises from a maximal lattice-free convex set M containing f and vice-verse. As this model has all possible columns and is effectively data independent, the result is 'cleaner' than the result of Theorem 3.3 and Theorem 3.4.

Theorem 3.5 has been recently generalized to the case where $W = \mathbb{R}^m$ by Basu et al. [10]. One of the significant difficulties in this generalization is proving that the minimal functions are nonnegative (in Section 2 the inequalities were assumed to have nonnegative coefficients. [10] does not make this assumption). We refer the readers to [10] for the details.

Addressing the case $m = 2$, Cornuéjols and Margot [15] described the extreme inequalities for the two-row set $R(f, \emptyset, \mathbb{Q}^2)$. In particular, if f lies in the interior of a split set, a maximal lattice-free triangle or a maximal lattice-free quadrilateral with no h satisfying (10), then π generated using (5) is an extreme inequality. Cornuéjols and Margot [15] also analyze the case of *degenerate* maximal lattice-free convex sets (i.e., sets containing f on the boundary) and show that even these sets can generate extreme inequalities.

Hence, when considering the case of a finite problem $R(f, \emptyset, W)$ ($r^j \in \mathbb{Q}^m$ for all $r^j \in W$ and $|W|$ is finite), the following question naturally arises: should one consider generating cuts using maximal lattice-free sets containing f on the boundary? For the case of two rows, Cornuéjols and Margot [15] show that none of the degenerate cases are needed to define the facets of $R(f, \emptyset, W)$. The same answer has been provided by Zambelli [38] for a general number of rows.

Theorem 3.6 ([38]). *Let $|W|$ be finite. Given a minimal valid inequality $\sum_{j=1}^{|W|} \alpha_j y_j \geq 1$ for $R(f, \emptyset, W)$ ($f, r^j \in \mathbb{Q}^m$), there exists a lattice-free convex set M such that f lies in its interior and the inequality generated using (5) satisfies $\alpha_j = \pi(r^j)$ for all j.*

3.4 Introducing Bounds on Nonbasic Variables

Andersen et al. [2] considered introducing upper bounds on the nonbasic variables for the two-row relaxation (4), i.e. they considered the set

$$\begin{aligned} &x_B = f + \textstyle\sum_{j=1}^k r^j y_j, \\ &x_B \in \mathbb{Z}^2, \quad y \in \mathbb{R}^k_+, \quad y_j \leq u_j \text{ for } j \in \{1, \ldots, k\} \setminus U, \end{aligned} \tag{12}$$

where $U \subseteq \{1, \ldots, k\}$ is the set of variables with no upper bound. In general, an inequality $\sum_{j=1}^k \alpha_j y_j \geq \alpha_0$ may have negative coefficients unlike in the previous sections. Andersen et al. [2] show that any non-trivial inequality for (12) can be written in the form $\sum_{j \in U} \alpha_j y_j + \sum_{j \in C_+} \alpha_j y_j + \sum_{j \in C_-} \alpha_j (u_j - y_j) \geq 1$ where $C_+ \cup C_- = \{1, \ldots, k\} \setminus U$ and each of the α_js are nonnegative.

An inequality $\sum_{j=1}^k \alpha_j y_j \geq \alpha_0$ is facet-defining for convex hull of (12) if and only if (α, α_0) is an extreme rays of the following polar cone:

$$\begin{aligned} \{(\alpha, \alpha_0) \in \mathbb{R}^{k+1} \,|\, &\alpha_j \geq 0 \ \forall j \in U \\ &\text{and } \sum_{j=1}^k \alpha_j y_j - \alpha_0 \geq 0 \ \forall y \in Y^v\}, \end{aligned} \tag{13}$$

where $Y^v := \{y \in \mathbb{R}^k_+ \,|\, \exists x_B \in \mathbb{Z}^2 \text{ s.t. } (x_B, y) \text{ is a vertex of conv } (12)\}$. Every extreme ray (corresponding to facet-defining inequality of (12)) can satisfy a large number of constraints of (13) at equality. For every extreme ray, Andersen et al. [2] present a non-trivial subset of inequalities that are tight at it and which define this facet-defining inequality uniquely. Using this result, Andersen et al. [2] are able to completely characterize facet-defining inequalities of (12) when exactly one of the nonbasic variables has an upper bound. The characterization is based on possible shapes of the set L_α (see (9)). Specifically, it is proven that L_α can take all the shapes presented in Theorem 3.3 along with pentagons. The pentagon represents a cut that is stronger than cuts that can be obtained without the information on the upper bound.

3.5 Introducing Constraints on Basic Integer Variables

Dey and Wolsey [21], Basu et al. [11], and Fukasawa and Günlük [25] have considered various variants of imposing constraints of the form $Ax_B \leq b$ on the basic integer variables of the model $R(f, \emptyset, W)$. This direction of research was previously investigated by Johnson [32].

Interestingly, many results in Section 3.2 and Section 3.3 carry through to this case. Observe that the nonnegativity constraints on the basic variable were removed to obtain the group relaxation. We then relaxed the integrality of nonbasic variables to obtain the relaxation $R(f,\emptyset,W)$ in Section 3.2. Now we are reintroducing the nonnegativity restriction (and more general constraints) on the basic variables.

In the case of the set $R(f,\emptyset,W)$, for any $\bar{x}_B \in \mathbb{Z}^m$ there exists a $\bar{y} \in \mathbb{R}_+^{|W|}$ such that $\bar{x}_B = f + \sum_{r^j\in W} r^j \bar{y}_j$ (assuming that $\text{cone}_{r^j\in W}\{r^j\} = \mathbb{R}^m$). Thus the intersection cut is generated using maximal lattice-free convex set. On the other hand, when we add constraints $Ax_B \le b$, instead of considering maximal lattice-free sets to generate intersection cuts we need to consider maximal convex sets M such that they contain no integer point satisfying $Ax_B \le b$ in their interior. Therefore, we now allow integer points satisfying $Ax_B > b$ in the interior of the convex set used to generate the cut. Formally we make the following definition as a counterpart to Definition 3.2.

Definition 3.3 ([21]). *Let $S \subseteq \mathbb{Z}^m$. A convex set $M \subseteq \mathbb{R}^m$ is a maximal S-free convex set if int$(M) \cap S = \emptyset$ and there exists no convex set M' such that int$(M') \cap S = \emptyset$ and $M' \supsetneq M$.*

It turns out that under the mild assumption of rationality of A and b, a result very similar to Theorem 3.2 carries through. Weaker versions of the following result are proven in Dey and Wolsey [21] and Fukasawa and Günlük [25].

Theorem 3.7 ([11]). *Let S be the set of integral points in some rational polyhedron in $\mathbb{R}^m$ such that* $\dim(\text{conv}(S)) = m$. *A set $M \subseteq \mathbb{R}^m$ is a maximal S-free convex set if and only if one of the following holds:*

1. *M is a polyhedron such that $M \cap \text{conv}(S)$ has nonempty interior, M does not contain any point of S in its interior and there is a point of S in the relative interior of each of its facets. The recession cone of $M \cap \text{conv}(S)$ is rational and it is contained in the lineality space of M.*
2. *M is a half-space of $\mathbb{R}^m$ such that $M \cap \text{conv}(S)$ has empty interior and the boundary of M is a supporting hyperplane of $\text{conv}(S)$.*
3. *M is a hyperplane of $\mathbb{R}^m$ such that $\text{lin}(M) \cap \text{rec}(\text{conv}(S))$ is not rational.*

Using maximal S-free convex sets, minimal inequalities can be generated. We consider the case when $W = \mathbb{R}^m$.

Theorem 3.8 ([21], [11]). *Let S be the set of integral points in some rational polyhedron in $\mathbb{R}^m$ such that* $\dim(\text{conv}(S)) = m$. *Let M be a maximal S-free convex set containing f in its interior. The set $M - \{f\}$ can be written in the form $\{x \mid (g^j)^T x \le 1, j \in \{1,\dots,l\}\}$ as $\bar{0}$ belongs to the interior of $M - \{f\}$ and M is polyhedral. Let*

$$\pi^M(u) = \max_{1\le j\le l}\{(g^j)^T u\}.$$

Then $\sum_{r^j\in\mathbb{R}^m} \pi^M(r^j) y_j \ge 1$ is a minimal inequality for the set $\{(x_B, y) \in R(f,\emptyset,\mathbb{R}^m) \mid x_B \in S\}$.

We note here that the inequality $\sum_{r^j\in\mathbb{R}^m} \pi^M(r^j) y_j \ge 1$ may have negative coefficients. Theorem 3.8 illustrates the use of maximal S-free convex sets to generate minimal inequalities. It is possible to construct maximal S-free convex set using minimal inequality (Theorem 3.9). This result together with Theorem 3.8 is a counterpart of Theorem 3.5. Basu et al. [11] present a very elegant proof of this result. (See [25] for the case of $m = 2$.)

Theorem 3.9 ([11]). *Let S be the set of integral points in some rational polyhedron in $\mathbb{R}^m$ such that* $\dim(\text{conv}(S)) = m$. *Let $\pi : \mathbb{R}^m \to \mathbb{R}$ be a minimal inequality for the set $\{(x_B, y) \in R(f,\emptyset,\mathbb{R}^m) \mid x_B \in S\}$ of the form $\sum_{r^j\in\mathbb{R}^m} \pi(r^j) y_j \ge 1$. Then the set $\{u \in \mathbb{R}^m \mid \pi(u - f) \le 1\}$ is a maximal S-free convex set.*

Dey and Wolsey [21] also characterize extreme inequalities for $\{(x_B, y) \in R(f,\emptyset,\mathbb{R}^2) \mid x_B \in S\}$. We refer the readers to [21] for the details.

3.6 Introducing Integral Nonbasic Variables

In Section 3.3, the integrality of nonbasic variables in the corner and group relaxation were relaxed to obtain the set $R(f,\emptyset,W)$. Now we consider reintroducing the integer nonbasic variables and study the set $R(f,I^m,\mathbb{R}^m)$. The focus is not on generating all possible minimal inequalities as characterized by Theorem 2.1. It is instead on generating first extreme inequalities for $R(f,\emptyset,\mathbb{R}^2)$ and then *lifting* in the integer nonbasic variables. One motivation for this approach is that the continuous variables get strongest possible coefficients in such a cut.

On the one hand, exact lifting of unbounded integer variables (or even nonnegative integer variables with upper bound greater than 1) is a difficult problem since it is a nonlinear integer program. On the other hand, a trivial valid inequality for $R(f,I^m,\mathbb{R}^m)$ is $\sum_{r^j\in I^m} \pi(r^j) x_j + \sum_{r^j\in\mathbb{R}^m} \pi(r^j) y_j \ge 1$ where the function π represents a valid inequality for $R(f,\emptyset,\mathbb{R}^2)$. The focus is to take a middle path, i.e. generate stronger coefficients without actually solving the exact lifting problem.

Dey and Wolsey [20, 23] considered the lifting of extreme inequalities for $R(f,\emptyset,\mathbb{R}^2)$ corresponding to maximal lattice-free splits, triangles and quadrilaterals using the so called trivial fill-in function. This approach coincides with a coefficient strengthening method presented by Balas and Jeroslow [8]. Given an extreme inequality for $R(f,\emptyset,\mathbb{R}^m)$, the trivial fill-in function $\phi^0 : I^m \to \mathbb{R}_+$, introduced by Gomory and Johnson [30], can be defined as $\phi^0(u) = \inf_{z\in\mathbb{Z}^m}\{\pi(u+z)\}$. Then, the following result holds.

Theorem 3.10 ([23]). *Let π be an extreme inequality for $R(f,\emptyset,\mathbb{R}^2)$. Then ϕ^0 is the unique lifting function in the case $L_\pi = \{u \in \mathbb{R}^2 \mid \pi(u - f) \le 1\}$ is a split set, a triangle of type 1, or a triangle of type 2 and (ϕ^0, π) is an extreme inequality for $R(f, I^2, \mathbb{R}^2)$. If $\{u \in \mathbb{R}^2 \mid \pi(u - f) \le 1\}$ is any other maximal lattice-free convex set, then there does not exist a unique lifting function and the trivial fill-in function is not minimal.*

Conforti et al. [13] have considered lifting of integer variables, starting from the minimal inequality for $\{(x_B, y) \in R(f,\emptyset,\mathbb{R}^m) \mid x_B \in S\}$.

Theorem 3.11 ([13]). *Let $\sum_{r^j\in\mathbb{R}^m} \pi(r^j) y_j \ge 1$ be a minimal inequality for $\{(x_B, y) \in R(f,\emptyset,\mathbb{R}^m) \mid x_B \in S\}$ where S is a set of integer points in a rational polyhedron. Let $\phi : \mathbb{R}^m \to \mathbb{R}$ be function such that (ϕ, π) is minimal for $\{(x_B, x, y) \in R(f,\mathbb{R}^m,\mathbb{R}^m) \mid x_B \in S\}$. Then there exists $\epsilon > 0$ such that $\pi(u) = \phi(u)$ for all $\|u\| \le \epsilon$.*

(Note that x_B variables are not free and two integer variables whose columns are equivalent modulo 1 may not get the same coefficient. Hence all possible columns are considered for non-basic integer variables.) Using Theorem 3.11, Conforti et al. [13] extend the results of Theorem 3.10 to higher dimensions and also present some other classes of inequalities for the set $\{(x_B, x, y) \in R(f,\mathbb{R}^m,\mathbb{R}^m) \mid x_B \in S\}$ that have unique lifting. We refer the readers to the paper [13] for details. Dey and Wolsey [22] have recently considered some mixed-lifting approaches combining traditional sequential lifting with the fill-in approach described above.

4 Properties of the New Cuts and Their Evaluation

4.1 Comparing Closures

There are a number of possible ways of evaluating the quality of the new classes of cutting planes that are derived from the extreme inequalities of multi-row relaxations. One approach is to compare the closures of various classes of inequalities. To this end, we require some definitions.

Given a polyhedron of the form $Q = \{x \in \mathbb{R}^n \mid a^i x \geq b^i, i = 1, \ldots, m\}$ where $a^i \geq 0$, $b^i \geq 0$ $\forall i = 1, \ldots, m$ and a scalar $\alpha > 0$, we define $\alpha Q = \{x \in \mathbb{R}^n \mid \alpha a^i x \geq b^i, i = 1, \ldots, m\}$. Note that $\alpha Q \supseteq Q$ where $\alpha \geq 1$. Larger the value of α, larger is the set αQ.

Given a class of cutting planes, the corresponding closure is defined as the set obtained by the addition of all possible inequalities of this class to the linear programming relaxation. Consider first the case of two rows. Let $R(f, \emptyset, W)$ where $r^j \in \mathbb{Q}^2$ for all $r^j \in W$, $f \in \mathbb{Q}^2$. Then the split closure $\mathcal{S}(f, \emptyset, W)$, the triangle closure $\mathcal{T}(f, \emptyset, W)$, and the quadrilateral closure $\mathcal{Q}(f, \emptyset, W)$ are obtained by intersecting the continuous relaxation of $R(f, \emptyset, W)$ with the cuts obtained using (5) where M is a all possible split sets, maximal lattice-free triangles, and maximal lattice-free quadrilaterals respectively. Since all the facet-defining (extreme) inequalities for $R(f, \emptyset, W)$ are either splits, triangles or quadrilaterals, we obtain conv$(R(f, \emptyset, W)) = \mathcal{S}(f, \emptyset, W) \cap \mathcal{T}(f, \emptyset, W) \cap \mathcal{Q}(f, \emptyset, W)$. Basu et al. [9] prove the following result.

Theorem 4.12 ([9]).

1. *Split versus triangle and quadrilateral closures:*
$$\mathcal{T}(f, \emptyset, W) \subseteq \mathcal{S}(f, \emptyset, W), \quad \mathcal{Q}(f, \emptyset, W) \subseteq \mathcal{S}(f, \emptyset, W).$$
2. *Triangle and quadrilateral closures versus conv$(R(f, \emptyset, W))$:*
$$\text{conv}(R(f, \emptyset, W)) \subseteq \mathcal{T}(f, \emptyset, W) \subseteq 2\text{conv}(R(f, \emptyset, W)),$$
$$\text{conv}(R(f, \emptyset, W)) \subseteq \mathcal{Q}(f, \emptyset, W) \subseteq 2\text{conv}(R(f, \emptyset, W)).$$
3. *Split closure versus conv$(R(f, \emptyset, W))$: For all $\alpha \geq 1$, there is a choice of W and f such that $\mathcal{S}(f, \emptyset, W) \nsubseteq \alpha\text{conv}(R(f, \emptyset, W))$.*

Theorem 4.12 proves that the split closure can be arbitrarily bad, while the triangle or quadrilateral closure get us within a factor of 2 of the convex hull no matter what the instance be. Therefore, at least theoretically, if we added all possible triangle or quadrilateral inequalities, the resulting set is quite a strong relaxation of conv$(R(f, \emptyset, W))$.

Andersen et al. [5] extended the result of [9] to the case of more rows. Let $\sum_{r^j \in W} \pi(r^j) y_j \geq 1$ be a valid inequality. Let d be the dimension of the linear space spanned by vectors r^j such that $\pi(r^j) = 0$. Then the *split-dimension* of the inequality $\sum_{r^j \in W} \pi(r^j) y_j \geq 1$ is defined as dim$(L_\pi) - d$ (see (9) for definition of L_π). Let $C^i(f, \emptyset, W)$ be the the intersection of all valid inequalities for conv$(R(f, \emptyset W))$ with a split dimension of at most i.

Theorem 4.13 ([5]). *For any $\alpha \geq 1$, there exist f, W ($f \in \mathbb{Q}^m, r^j \in \mathbb{Q}^m$ for all $r^j \in W$, $|W|$ finite) such that $C^{m-1}(f, \emptyset, W) \nsubseteq \alpha\text{conv}(R(f, \emptyset, W))$.*

It is well known that the split closure of any mixed integer set is a polyhedron (Cook et al. [14]). It would interesting to obtain similar results or to provide counterexample for the new class of inequalities. Andersen et al. [1] provide some answers in this direction. We first require one definition. Given a facet $g^T x \geq g^0$ of a lattice-free convex set M, let $w(g, M) := \max_{x \in M} g^T x - \min_{x \in M} g^T x$. The *facet-width* of M is defined as the maximum of $w(g, M)$ over all facets of M. Let $\mathcal{P} \subseteq \mathbb{Z}^m \times \mathbb{R}^n$ be a mixed integer linear set. Let M be lattice-free convex sets in m-dimensions and let $M' = \{(u, v) \in \mathbb{R}^m \times \mathbb{R}^n \mid u \in M\}$. As discussed in Section 3.1, the linear programming relaxation of $\mathcal{P}$ (denoted $\mathcal{P}^0$) can be strengthened by computing $U(\mathcal{P}, M) := \text{conv}(\mathcal{P}^0 \setminus \text{int}(M'))$. Define the $w-$split closure of $\mathcal{P}$ to be the set $\cap_{\text{facet-width } of\, M \leq w} U(\mathcal{P}, M)$.

Theorem 4.14 ([1]). *The $w-$split closure of $\mathcal{P}$ is a polyhedron.*

4.2 Split Rank

Another approach to compare the new families of cutting planes with split cuts is to determine the *split rank* of the new families of cutting planes. As discussed in Section 4.1 the split closure is a polyhedral relaxation of the convex hull of a mixed integer set. It is possible to apply the split closure procedure to the the resulting polyhedral relaxation to obtain the second split closure. Inductively, we define the k^{th} split closure as the split closure of the $(k-1)^{\text{th}}$ split closure. The split rank of an inequality is defined as the smallest integer k such the inequality is valid for the k^{th} split closure. Thus, the cutting plane (8) in Example 3.1 does not have a finite split rank since it cannot be obtained in a finite number of split closure procedure. As presented next, Dey and Louveaux [17] show that this is the 'only' interesting example with this property for the two-row case.

Theorem 4.15 ([17]). *Let $\sum_{r^j \in W} \pi(r^j) y_j \geq 1$ be a facet-defining inequality for $R(f, \emptyset, W)$ ($f, r^j \in \mathbb{Q}^2$, $|W|$ is finite). Then the split rank of $\sum_{r^j \in W} \pi(r^j) y_j \geq 1$ is finite if and only if L_π is not a maximal lattice-free triangle of type 1.*

For the general case of m rows, Dey [16] presents a geometric argument to determine a lower bound on the split rank of intersection cuts applied to a mixed integer set of the form $R(f, \emptyset, W)$ ($f \in \mathbb{Q}^m, r^j \in \mathbb{Q}^m$ for all $r^j \in W$, $|W|$ finite): Given the inequality $\sum_{r^j \in W} \pi(r^j) y_j \geq 1$, first a polyhedral subset of L_π called restricted lattice-free convex is constructed (under a technical assumption on the columns W). Then it is shown that $\lceil \log_2(l) \rceil$ is a lower bound on the split rank of the intersection cut where $\{x^1, x^2, \ldots, x^l\}$ is a subset of integer points on the boundary of the restricted lattice-free set such that no two points lie on the same facet of the restricted lattice-free set. We refer the readers to [16] for details.

4.3 Computational Experiments

A first computational investigation to understand the practical impact of multi-row cuts has been conducted by Espinoza [24]. In [24], multi-row cuts are embedded in CPLEX 10.2 default branch-and-cut by using CPLEX callbacks and are separated at the root node of the branch-and-cut tree, after CPLEX default cutting planes. These cuts are generated by relaxing the simplex tableau of a general MIP as a set of the form $R(f, \emptyset, W)$. Espinoza [24] considered sets with m varying from 2 up to 15, and two classes of maximal lattice-free bounded convex sets for generating cuts using (5). The first class of maximal lattice-free convex set is a special simplex, while the second one is the cross polytope having 2^m facets.

The computational experiments provided by Espinoza [24] compare CPLEX branch-and-cut enforced with multi-row cuts versus CPLEX default on a test bed of 87 MIPLIB 3.0 and MIPLIB 2003 instances. Despite some negative examples in which CPLEX default yields a better dual bound at the root node and solves the problem faster, the reported results are overall encouraging, thus showing that the use of multi-row cutting planes may hold some potential. A geometric average speed-up over CPLEX default of 31% can be observed for instances in which optimality is reached using the additional cuts. Finally, as reported in [24], it is worth noting that the performance of the multi-row cuts seems to improve when these cuts are based on a larger set of rows, since the integrality gap closed at the root node tends to increase, while a smaller number of cuts is typically generated.

5 Open Questions

On the theoretical side, there are a large number of open questions. We mention two here. Maximal lattice-free convex sets in $\mathbb{R}^2$

are well understood. In $\mathbb{R}^3$, some properties of maximal lattice-free convex sets are known (see Scarf [35], Andersen et al. [4]). However, not much is known about maximal lattice-free convex sets in higher dimensions. Another promising direction of research is to understand how many of the relaxations considered to construct the set $R(f, \emptyset, W)$ can be revoked, while still having the possibility of providing a complete characterization of all extreme inequalities.

The most significant challenges are probably on the practical side. While Espinoza [24] presents encouraging result, there are numerous avenues for improvement. The primary difficulty is that of cut selection. Consider the case of two rows: the number of possible choices of rows is $\mathcal{O}(m^2)$ where m is the number of rows. For each choice of two rows, there are then a number of possibilities in term of selection of triangles or quadrilaterals. With so many cuts, an appropriate tool for cut selection is vital for a successful implementation. Another practical difficulty with this class of cutting planes is that they tend to be dense. This can affect the performance of other components of a branch-and-cut algorithm. At this time various computational experiments are underway and we hope that much progress will be made in terms of successfully exploiting these cutting planes.

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Discussion Column

Gérard Cornuéjols, Robert Weismantel and Laurence Wolsey

Comments on Multi-Row Cuts

Gérard: I find the recent investigations on multi-row cuts fascinating. What really got me hooked was the paper that the two of you wrote with Kent Andersen and Quentin Louveaux [2]. Kent showed me a preliminary draft of the paper at the Math Programming Symposium in Rio in 2006 and I really liked the approach. I felt that it brought a new dimension to Integer Programming. It was a big departure from recent investigations in the theory of cutting planes, such as the study of mixed integer cuts which can be generated from integrality arguments applied to a single equation. Of course there are connections between your work and that of Egon Balas on intersection cuts generated from convex sets in the 1970s, or of Ralph Gomory and Ellis Johnson on the corner polyhedron around the same period. A great novelty of your paper was to consider a model where integrality only appears in the basic variables. This is a very appealing model, which preserves much of the complexity of Integer Programming and at the same time is sufficiently simplified that one can prove a lot about it. In particular, only very special convex sets can give rise to facets in this model. The paper that you wrote with Kent and Quentin studies the 2-row case, and you showed that only nonnegativity, split, triangles and quadrilaterals generate facets. What an elegant result! What got you interested in this line of research, Laurence? What about you Robert?

Laurence: Hard to remember. However I, like Gérard, was frustrated by the single row case in which he and others had failed to produce anything stronger than the Gomory Mixed Integer cuts. I was intrigued by the Cook-Kannan-Schrijver example with infinite convergence [4], and for years the question of handling two rows whether 0-1 knapsacks or other, was a problem that I kept coming back to. I also was taken by the computational work of Kent Andersen, Yanjun Li and Gérard [1] in which they heuristically combined rows so as to get cuts with small coefficients on the continuous non-basic variables.

So when Kent came to Core, we almost immediately decided to look a the two row group problem. What was interesting was the fact that the problem with just two continuous variables was equivalent to a two variable IP which meant there were compact extended formulations for the problem with continuous non-basics. Also the Cook-Kannan-Schrijver example fits the two constraint case. We soon found ourselves looking at lattice-free triangles (back to Balas intersection cuts and Ellis Johnson's gauge functions), and managed to generate some strong valid inequalities. However the real progress was then made by Kent, Robert and Quentin in Magdeburg.

Later when Santanu Dey came to Core, my main concern was how to deal with integer non-basic variables, which again led him far beyond what was initially asked.

Robert: My point of departure was that in the mixed-integer setting no generally finite cutting plane algorithm is known. In particular, the paper [4] shows that by using regular split cuts one cannot always terminate in finite time, even not for problems with two integer variables and one continuous one. The model that we introduced in [2] allows us to generate a triangle cut that leads to finite termination of a cutting plane algorithm for those low dimensional examples. From my perspective, the important feature of the model is the correspondence between facets for such models and maximally lattice-point-free polyhedra. This link has been established by now quite well in a series of follow-up papers such as [5], [7] and extensions by you, Gérard, Michele Conforti, Giacomo Zambelli, Amitabh Basu and further papers by Santanu Dey and Laurence or by Kent Andersen, Christian Wagner and myself. I would like to comment a bit further on this topic, namely the link between cutting planes and lattice-point-free polyhedra. In fact, for any polyhedron L with no interior integer points and any polyhedron P we can generate inequalities that are valid for the mixed integer points in P by taking the convex hull of all points that are in P, but not in the interior of L. In order to generate strong cuts this way we rather make L as large as possible (Maximally lattice-point-free). Every maximally lattice-point-free polyhedron is the Minkowski sum of a polytope and a linear space whose dimension is equal to the codimension of the polytope part. Clearly, the complexity of the cuts increases with the dimension of this polytope part. Indeed, it was shown by Jörg [11] that if one allows for arbitrary disjunctions, then a finite cutting plane algorithm for mixed integer optimization can be developed. It remains, however, open to understand precisely the connection between the geometry of the lattice-point-free polyhedron and the event of finite termination. More precisely, suppose that one starts with cuts arising from parallel splits. After a certain number of rounds it is expected that the progress in terms of improvement of the objective function becomes small (tailing off). Once the improvement in terms of the objective function is below a certain threshold, disjunctions based on two –and even higher dimensional lattice-point-free bodies– should be applied. Which conditions must such a body satisfy in order to avoid this tailing off effect? Answers to this question will allow us to design a hierarchical scheme for using more and more complex lattice-point-free polyhedra as a cut generation machine. Finally it remains to comment on the question how far this approach will take us in terms of computations. I am very optimistic and hope to share some promising experimental results with all of you by the end of the year. Kent Andersen, Christian Wagner and I are currently working on that.

Gérard Thanks for the insights into what motivated you both to studying 2-row cuts and why this area is so exciting! There are two key relaxations in the model you proposed in [2]. One is relaxing integrality of the nonbasic variables and the other is relaxing nonnegativity of the basic variables. The latter relaxation is the classical group relaxation of Gomory. The first is what really appealed to me and drew me into studying this area myself. A natural question is: What happens if one only makes one of these two relaxations? Can one say anything interesting building on the results of [2]? Laurence and Santanu have looked at both strengthenings of the model in [2] and came up with two great papers [9], [10]. The first paper lifts minimal inequalities from [2] into valid inequalities for the group problem and shows that in several interesting cases the minimal lifting function is unique. This is important work. But I find the second paper even more surprising. A development that I did not anticipate is that much of the theory in [2] and subsequent papers goes through even when only the first relaxation occurs. In other words, the key to the beautiful connection to maximal lattice-free convex sets is not the group relaxation but rather relaxing integrality of the nonbasic variables. This was nicely brought out in a paper that Laurence and Santanu wrote this Spring [10]. There are new difficulties since now facets can have negative coefficients and the connection to maximal lattice-free convex sets is not quite so simple. But most of the theory goes through. To me this means that the greatest insights come not from the group relaxation but from the other relaxation, namely relaxing integrality of the nonbasic variables. No doubt this is a controversial statement. Any comments?

Laurence I like Gérard's viewpoint, though it may just be a case of transferring difficulties elsewhere. In particular Michele, Gérard and Giacomo [6] have just shown the generality of this model. By augmenting the number of rows and columns, they show how to reformulate the group relaxation as an equivalent problem with integer basic variables under some additional constraints and nothing but real-valued nonnegative non-basic variables.

There are several other intriguing questions. Santanu Dey and Quentin Louveaux [8] have shown that for two row problems the only valid inequalities that have infinite split rank are those arising from the triangles with integer vertices corresponding to the Cook-Kannan-Schrijver example. Do maximal lattice-free bodies, studied by Kent Andersen, Christian Wagner and Robert [3] also play a special role in higher dimensions? Does this mean that in the 2-row case a cutting plane algorithm based on Gomory mixed integer cuts solves finitely for all other objective functions?

Another wide open question concerns the links between branching and cutting, see the experiments of Miroslav Karamanov and Gérard [12] and the work of Jörg cited above. Should we be using three way branching rather than trying to use inequalities based on lattice-free triangles, etc.?

Finally the number of computational options for generating facets off relaxations from two or more rows are immense, so we wait with baited breath for the magic recipe. There's still much work here for everyone.

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Prizes Presented at ISMP

George B. Dantzig Prize

Jointly awarded by the MPS and the Society of Industrial and Applied Mathematics (SIAM). Committee: Yurii Nesterov, Jong-Shi Pang (Chair), Lex Schrijver, Eva Tardos.

The George B. Dantzig prize is awarded for original research, which by its originality, breadth and scope, is having a major impact on the field of mathematical programming. The winner is Professor Gérard Cornuéjols from Carnegie-Mellon University, for his deep and wide-ranging contributions to mathematical programming, including his work on Balanced and Ideal Matrices and Perfect Graphs and his leading role in the work on general cutting planes for mixed integer programming over many years covering both theory and computation.

The Beale-Orchard-Hays Prize

Committee: Erling Andersen, Philip Gill, Jeff Linderoth, Nick Sahinidis (chair).

This Prize is sponsored by the Society in memory of Martin Beale and William Orchard-Hays, pioneers in computational mathematical programming. The Prize is given for excellence in any aspect of computational mathematical programming. 'Computational mathematical programming' includes the development of high-quality mathematical programming algorithms and software, the experimental evaluation of mathematical programming algorithms, and the development of new methods for the empirical testing of mathematical programming techniques.

Selecting a prize winner this year was challenging since we received thirteen exceptional nominations. The nominated works spanned a broad spectrum of areas, including linear programming, semidefinite programming, nonlinear programming, integer programming, stochastic programming, and global optimization. In reflection of the Mathematical Programming Society's international character, these nominations originated from ten different countries.

The 2009 Prize was awarded to Tobias Achterberg for his paper "SCIP: Solving constrained integer programs," *Mathematical Programming Computation*, 1 (2009), pp. 1–41, which is the first paper to appear in the Society's new journal *Mathematical Programming Computation*. This paper describes an innovative paradigm to integrate modeling capabilities and solution techniques from constraint programming, mixed-integer programming, and satisfiability. These techniques are carefully integrated after extensive computational experimentation that resulted in the software SCIP, which consists of more than 250,000 lines of code, all written by the author himself, and which is the focus of the paper. The source code of SCIP is available free for academic use. Compared to previous branch-and-bound systems, SCIP achieves a tighter integration of the aforementioned modeling and solution paradigms. In addition, it implements several novel algorithmic constructs that have been developed in recent papers by the author and co-workers, including branching rule scores, cutting plane handling, and conflict analysis. It is truly remarkable that SCIP has performed with times that are within a modest factor of those for the best commercial codes for mixed-integer programs. This fact alone is one of the best confirmations of the quality of the work done by the author. An additional strength of SCIP is the ease with which researchers can create branch-cut-and-price applications by implementing "plug-ins" for the problem-specific routines. Finally, the computational results in the paper show that the approach developed by the author outperforms current state-of-the-art techniques for proving the validity of properties on circuits containing arithmetic.

Fulkerson Prize

Jointly awarded by the MPS and the American Mathematical Society (AMS). Committee: William Cook (chair), Michel Goemans, Daniel Kleitman.

> Maria Chudnovsky, Neil Robertson, Paul Seymour, and Robin Thomas, The strong perfect graph theorem, *Annals of Mathematics* **164** (2006) 51–229.

Claude Berge introduced the class of perfect graphs in 1960 together with a possible characterization in terms of forbidden subgraphs. The resolution of Berge's strong perfect graph conjecture quickly became one of the most sought after goals in graph theory. The pursuit of the conjecture brought together four important elements: vertex colorings, stable sets, cliques, and clique covers. Moreover, D. R. Fulkerson established connections between perfect graphs and integer programming through his theory of anti-blocking polyhedra. A graph is called perfect if for every induced subgraph H the clique-covering number of H is equal to the cardinality of its largest stable set. The strong perfect graph conjecture states that a graph is perfect if and only if neither it nor its complement contains as an induced subgraph an odd circuit having at least five edges. The elegance and simplicity of this possible characterization led to a great body of work in the literature, culminating in the Chudnovsky et al. proof, announced in May 2002, just one month before Berge passed away. The long, difficult, and creative proof by Chudnovsky et al. is one of the great achievements in discrete mathematics.

Thomas C. Hales, A proof of the Kepler conjecture, *Annals of Mathematics* **162** (2005) 1063–1183.
Samuel P. Ferguson, Sphere Packings, V. Pentahedral Prisms, *Discrete and Computational Geometry* **36** (2006) 167–204.

In 1611 Johannes Kepler asserted that the densest packing of equal-radius spheres is obtained by the familiar cannonball arrangement. This statement is known as the Kepler conjecture and it is a component of Hilbert's 18th problem. After four centuries, Ferguson and Hales have now proven Kepler's assertion. The Ferguson-Hales proof develops deep connections between sphere packings and mathematical programming, making heavy use of linear programming duality and branch and bound to establish results on the density of candidate configurations of spheres. The beautiful geometric arguments and innovative use of computational tools make this a landmark result in both geometry and discrete mathematics.

Daniel A. Spielman and Shang-Hua Teng, Smoothed analysis of algorithms: Why the simplex algorithm usually takes polynomial time, *Journal of the ACM* **51** (2004) 385–463.

George Dantzig's simplex algorithm for linear programming is a fundamental tool in applied mathematics. The work of Spielman and Teng is an important step towards providing a theoretical understanding of the algorithm's great success in practice despite its known exponential worst-case behavior. The smoothed analysis introduced by the authors fits nicely between overly pessimistic worst-case results and the average-case theory developed in the 1980s. In smoothed analysis, the performance of an algorithm is measured under small perturbations of arbitrary real inputs. The Spielman-Teng proof that the simplex algorithm runs in polynomial-time under this measure combines beautiful technical results that intersect multiple areas of discrete mathematics. Moreover, the general smoothed analysis framework is one that can be applied in many algorithmic settings and it is now established as an important technique in theoretical computer science.

Lagrange Prize

Jointly awarded by the MPS and the SIAM. Commitee: Adrian Lewis (Chair), Jorge Moré, Philippe Toint, Margaret Wright.

Jean B. Lasserre: A sum of squares approximation of nonnegative polynomials, *SIAM Journal on Optimization* **16** (2006), 751–765. (The paper was also reproduced in the SIGEST section of *SIAM Review* **49** (2007), 651-669.)

Lasserre has been a pioneer in the field of global polynomial optimization since his 2001 paper "Global optimization with polynomials and the problem of moments", SIOPT 11, 796–817. His approach relies on certificates of positivity for polynomials on compact sets defined by polynomial inequalities, using representations as sums of squared polynomials - a deep and powerful technique taken from real algebraic geometry. Using semidefinite programming to recognize sums of squares, Lasserre constructs a hierarchy of semidefinite relaxations converging to a solution of the (NP hard!) global optimization problem. The convergence is often fast or even finite in practice, as illustrated in software packages such as GloptiPoly, SOSTOOLS and SPARSEPOPT. The duality theory for the semidefinite relaxations corresponds exactly to the duality between polynomial optimization and generalized moment problems, a relationship explored in great generality in Lasserre's paper "A semidefinite programming approach to the generalized problem of moments", MPB 112 (2008), 65–92.

The winning paper is a particularly striking exemplar of Lasserre's work on polynomial optimization. He presents an elegant new proof of a classical foundation stone for the theory: that any nonnegative polynomial can be approximated by sums of squares. Combining convex duality theory and a moment-theoretic result, he constructs approximating polynomials that are both simple and explicit. One consequence is a satisfying simplification of his original relaxation hierarchy. The paper is a beautiful blend of modern optimization theory and deep classical mathematics, with striking computational implications.

Tucker Prize

Committee: Frederic Bonnans, Fritz Eisenbrand, Sven Leyffer, Franz Rendl (chair), Ruediger Schultz.

The Tucker Prize for an outstanding paper authored by a student has been awarded to Mohit Singh, Microsoft Research, Cambridge, for his thesis: *Iterative Methods in Combinatorial Optimization*.

Mohit Singh obtained his undergraduate degree in Computer Science and Engineering from the Indian Institute of Technology, Delhi in 2003. He completed his Ph.D. in the Algorithms, Combinatorics and Optimization program from Tepper School of Business, CMU under the supervision of Prof. R. Ravi in May 2008. He is currently a post-doctoral researcher at Microsoft Research, New England. His research interests lie in theoretical computer science, combinatorial optimization, and approximation algorithms. He published two or more papers in each of the following prestigious conferences STOC, FOCS, IPCO and APPROX.

Singh's thesis introduces a new technique for solving important optimization problem exactly and extends it to solving their NP-hard variants obtained by introducing complicating side constraints.

The method, iterative in nature, suggests a natural recursive algorithm for obtaining exact solutions to problems by writing a carefully constructed linear programming relaxation and arguing that the relaxation always has a zero- or one-element in an optimal basic solution. The crux of the argument exploits the fact that the set of tight constraints at a basic solution can be represented by a sparse family, hence the cardinality of the support of the solution is also sparse.

The thesis shows a large set of problems amenable to this technique ranging from spanning trees in directed and undirected graphs, minimal matroid bases and perfect matchings.

While it is an impressive contribution to offer novel proofs of several classical polyhedral results with implications to exact algorithms, the thesis makes an even larger contribution by showing how the method can be extended to handle side constraints.

As an example, consider the classical Minimum Spanning Tree problem with upper bounds on the degrees of the various nodes in the tree. If the current fractional support of nonzero edges have degree at most one more than the degree bound at some node, the degree constraint for this node can be dropped since in the worst case, even if all these edges in the fractional support were chosen in the final solution, the degree constraint will only be violated by one. Singh's striking result (obtained jointly with Lau) shows that if there are no zero- or one-valued edges, there will always be such a degree constraint on some node that one can drop and continue iteratively looking for more zero- or one-edges (to delete or include) or more constraints to relax with low violation. This settles a long-standing conjecture for this problem due to Goemans affirmatively.

Singh's thesis carefully rounds out the various other classes of optimization problems where such side constraints can be handled approximately; He derives new approximation results for general connectivity problems (such as survivable network design) with side degree constraints, as well as for multicriteria spanning tree and matroid bases problems. The thesis also gives new short proofs of various constrained optimization problems such as the generalized assignment problem using the same relaxation framework.

Singh's thesis research has already attracted follow up work in several prestigious conferences such as STOC 2008, IPCO 2008 and FOCS 2008, and continues to generate a flurry of research activity around new applications of these techniques.

This is a very impressive dissertation, which by its breath and depth qualifies the work as the winner of the 2009 A.W. Tucker Prize.

The other two Tucker Prize finalists chosen by this year's Tucker Prize Committee are Tobias Achterberg and Jiawang Nie.

Tobias Achterberg studied at the Technical University of Berlin where he finished both his master and his doctoral studies. The title of his dissertation is: *Constraint Integer Programming*. It was supervised by Martin Grötschel, and finished during the summer of 2007. He is currently with IBM-CPLEX as a software developer.

In his thesis, Achterberg discusses the integration of techniques from mixed-integer programming (MIP), constraint programming (CP), and satisfiability (SAT) solving. All three areas deal with optimization or feasibility problems over integer variables, which can be solved by tree search algorithms. Numerous industrial applications can be modeled as MIP, CP, or SAT. In particular, MIP has drawn a lot of academic and commercial attention.

Achterberg introduces the concept of constraint integer programming (CIP), which generalizes mixed-integer programming in order to carry over the powerful modeling techniques from constraint programming to mixed-integer programming. He makes use of the entire theory of constraint and integer programming and compares their theoretical advantages and disadvantages. He analyzes the different techniques carefully by conducting practical experiments, and he implements all variants with outstanding program quality and remarkable attention to detail. This results in the software SCIP, which consists of more than 250,000 lines of code. Although CIP is a more general problem class than MIP, the performance of SCIP on pure MIP instances is comparable to the best commercial MIP codes.

By now, the code of Achterberg is well-known and recognized in the academic world. Independent researchers identified it as the best non-commercial code for MIP, and it is used in a variety of academic and industrial projects. The fact that the source code of the software is freely available opens up new research possibilities in the area of optimization.

Jiawang Nie did his undergraduate studies in China, finishing with a master of science in 2000 at the Chinese Academy of Sciences. He then moved to the University of California, Berkeley, where he wrote a dissertation entitled: *Global Optimization of Polynomial Functions and Applications*, supervised by James Demmel and Bernd Sturmfels. The thesis was finished in the fall of 2006. Jiawang Nie is currently assistant professor at the University of California in San Diego.

Nie's dissertation focusses on the interplay between optimization with polynomial functions and semidefinite programming (SDP). More precisely, he showed that quite general (and extremely difficult) problems in polynomial optimization could be solved by a converging sequence of approximations, each efficiently computable using recently developed techniques of SDP, and he formulated and solved engineering design optimization problems using these techniques.

One key ingredient lies in the observation that a polynomial is necessarily nonnegative if it has a 'sums of square' (SOS) representation through other polynomials. In the dissertation he explores this idea applied both in the context of minimizing polynomials via Sum of Squares over the Gradient Ideal, and also through representations of positive polynomials on non-compact semialgebraic sets via Karush-Kuhn-Tucker ideals. His work improves on previous results of Lasserre, Jibetean, Laurent and Parrilo by removing assumptions such that the gradient variety must be finite. By exploiting the Kuhn-Karush-Tucker condition, he extends his results from unconstrained to constrained optimization. These are interesting and far reaching results, and make a significant improvement over the prior best results in this area.

These theoretical results are successfully applied in various areas of engineering, for instance shape optimization, perturbation analysis of polynomial systems of equations, or sensor network location.

Nie currently has more than a dozen publications in top quality journals on optimization, such as Mathematical Programming and SIOPT.

Ignacio E. Grossmann and Jon Lee

New MINLP Cybersite

In a joint collaboration between Carnegie Mellon University and the IBM T. J. Watson Research Center, researchers have launched a CyberInfrastructure Collaborative site for Mixed-Integer Nonlinear Programming (MINLP): www.minlp.org, that has been funded by the National Science Foundation under Grant OCI-0750826: "Open CyberInfrastructure for Mixed-integer Nonlinear Programming: Collaboration and Deployment via Virtual Environments." The core team consists of: Larry Biegler, Ignacio E. Grossmann, François Margot and Nick Sahinidis of CMU, and Jon Lee and Andreas Wächter of IBM. Additional collaborators include: Pietro Belotti (Lehigh University), Pedro Castro (INETI) and Juan Ruiz (CMU).

The major goal of this site is to create a library of optimization problems in different application areas in which one or several alternative models are presented with the derivation of their mathematical formulations. In addition, each model has one or several instances that can serve to test various algorithms. While we are emphasizing MINLP models, you may also wish to submit MILP and NLP models that are particularly relevant to problems that also have MINLP formulations. The site is intended to provide a mechanism for researchers and users to contribute towards the creation of the library of optimization problems, and to provide a forum of discussion for algorithm developers and application users where alternative formulations, as well as performance and comparison of algorithms can be discussed. The site also provides information on various resources, meetings and a bibliography.

We are looking for new contributions to expand the library of test problems, and also for feedback on this site. Comments are welcome at: minlp@andrew.cmu.edu.

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