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Mathematical Optimization Society Newsletter

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MOS Chair's Column

March 30, 2011. The Mathematical Optimization Society is alive and active, and the best proof of this statement is the number of innovations that have taken place within the Society since I wrote my last column in Optima. Let me review these innovations briefly.

The first innovation, especially from the researcher point of view, is the new internet support which has been introduced for our main scientific journal, *Mathematical Programming (Series A)*. This change (described in another column in this issue of Optima) aims at providing better service to the optimization research community, and, I am certain, will be beneficial for this community.

The second important event I wish to bring to your attention is the creation of "Named Lectureships" at the Mathematical Programming International Symposium. These lectureships are intended to honour prominent researchers in optimization and related fields by selecting a scientist to give a special lecture during the Symposium (and possibly attributing a cash prize associated with the lectureship). Fittingly, the first such named lectureship, which has been accepted by the Council of the Society, is the "Paul Y. Tseng Memorial Lectureship in Continuous Optimization". It will be presented for the first time at the Twenty First International Symposium of Mathematical Programming (ISMP) in 2012, and triennally at each ISMP thereafter. This lectureship was established on the initiative of family and friends of Professor Tseng, with financial contributions to the associated endowment also coming from universities and companies in the Asia Pacific region. The purposes of the lectureship are to commemorate the outstanding contributions of Professor Tseng in continuous optimization and to promote the research and applications of continuous optimization in the Asia Pacific region. The Council of the Society has not excluded that other named lectureships could be established in the future, and has modified the bylaws of the Society (available on the MOS website) to establish suitable rules and conditions.

Needless to say, I am extremely pleased and proud of this new valuable development, and I am truly looking forward to the first Paul Tseng Lecture in Berlin.

A third potentially important decision has been taken by the MOS Council regarding the collection of the Society's membership fees. It is a very unfortunate fact that so far the MOS membership falls significantly once the memberships granted during the International Symposium expire. As it turns out, many members forget to renew their membership and this puts the whole Society in a difficult situation, because this happens in the period during which the next Symposium is being organized. During this period a healthy membership is needed to give the organizers a stronger negotiating position with local facilitators, such as hotels or universities. The council has therefore decided to propose financially attractive multi-year membership packages during the registration process of the Symposium. The first such offer will be made available at the registration for the Berlin 2012 Symposium.

This last topic provides the necessary transition for my (admittedly repeated) urgent call to all: *Please consider being a MOS member*, or to renew your membership for 2011 if you have not yet done so. Please suggest MOS membership to your colleagues and students. We need a strong society to represent our research community and help the organizers of the fantastic forthcoming Berlin Symposium.

Note from the Editors

Dear MOS Members,
We are pleased to take over the editorship of the Optima newsletter from the extremely capable team of editor Andrea Lodi and co-editors Alberto Caprara and Katya Scheinberg (who has been promoted to editor). We thank them for their tremendous service!

We want to briefly introduce the new team, which is as geographically and scientifically diverse as ever. Katya Scheinberg (katyas@lehigh.edu) works in various areas of continuous optimization with special interests in derivative free optimization and applications of statistical learning. Sam Burer (samuel-burer@uiowa.edu) conducts research in convex optimization and strives to write fast, accurate optimization code. Volker Kaibel's (kaibel@ovgu.de) main scientific interest is in discrete optimization with special emphasis on polyhedral aspects.

As the editorial board, our hope is to continue Optima as a timely newsletter with high quality contributions. We will retain the format established by the previous board with one main scientific contribution per issue and a discussion column on a related topic. We hope that the readers will continue finding this format stimulating. Please let us know if you have a contribution yourself or if there is a particular topic you would like to see covered in Optima.

In this issue we present an article by our own new co-editor, Volker Kaibel, and two discussion pieces all on the topic of extended formulations.

Katya Scheinberg, Editor
Sam Burer, Co-Editor
Volker Kaibel, Co-Editor

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Volker Kaibel

Extended Formulations in Combinatorial Optimization

1 Introduction

Linear Programming based methods and polyhedral theory form the backbone of large parts of Combinatorial Optimization. The basic paradigm here is to identify the feasible solutions to a given problem with some vectors in such a way that the optimization problem becomes the problem of optimizing a linear function over the finite set X of these vectors. The optimal value of a linear function over X is equal to its optimal value over the convex hull $\text{conv}(X) = \{\sum_{x\in X} \lambda_x x : \sum_{x\in X} \lambda_x = 1, \lambda \geq \mathbb{O}\}$ of X. According to the Weyl–Minkowski Theorem [33, 25], every *polytope* (i.e., the convex hull of a finite set of vectors) can be written as the set of solutions to a system of linear equations and inequalities. Thus one ends up with a linear programming problem.

As for the maybe most classical example, let us consider the set $\mathcal{M}(n)$ of all matchings in the complete graph $K_n = (V_n, E_n)$ on n nodes (where a matching is a subset of edges no two of which share a common end-node). Identifying every matching $M \subseteq E_n$ with its characteristic vector $\chi(M) \in \{0,1\}^{E_n}$ (where $\chi(M)_e = 1$ if and only if $e \in M$), we obtain the *matching polytope* $\mathrm{P}^{\text{match}}(n) = \text{conv}\{\chi(M) : M \in \mathcal{M}(n)\}$. In one of his seminal papers, Edmonds [13] proved that $\mathrm{P}^{\text{match}}(n)$ equals the set of all $x \in \mathbb{R}^{E_n}_+$ that satisfy the inequalities $x(\delta(v)) \leq 1$ for all $v \in V_n$ and $x(E_n(S)) \leq \lfloor |S|/2 \rfloor$ for all subsets $S \subseteq V_n$ of *odd* cardinality $3 \leq |S| \leq n$ (where $\delta(v)$ is the set of all edges incident to v, $E_n(S)$ is the set of all edges with both end-nodes in S, and $x(F) = \sum_{e\in F} x_e$). No inequality in this system, whose size is exponential in n, is redundant.

The situation is quite similar for the *permutahedron* $\mathrm{P}^{\text{perm}}(n)$, i.e., the convex hull of all vectors that arise from permuting the components of $(1, 2, \ldots, n)$. Rado [29] proved that $\mathrm{P}^{\text{perm}}(n)$ is described by the equation $x([n]) = n(n+1)/2$ and the inequalities $x(S) \geq |S|(|S|+1)/2$ for all $\emptyset \neq S \subsetneq [n]$ (with $[n] = \{1, \ldots, n\}$), none of the $2^n - 2$ inequalities being redundant. However if for each permutation $\sigma : [n] \to [n]$ we consider the corresponding *permutation matrix* $y \in \{0,1\}^{n\times n}$ (satisfying $y_{ij} = 1$ if and only if $\sigma(i) = j$) rather than the vector $(\sigma(1), \ldots, \sigma(n))$, we obtain a much smaller description of the resulting polytope, since, according to Birkhoff [7] and von Neumann [32], the convex hull $\mathrm{P}^{\text{birk}}(n)$ (the *Birkhoff-Polytope*) of all $n \times n$-permutation matrices is equal to the set of all *doubly-stochastic* $n \times n$-matrices (i.e., nonnegative $n \times n$-matrices all of whose row- and column sums are equal to one). It is easy to see that the permutahedron $\mathrm{P}^{\text{perm}}(n)$ is a linear projection of the Birkhoff-polytope $\mathrm{P}^{\text{birk}}(n)$ via the map defined by $p(y)_i = \sum_{j=1}^n j y_{ij}$. Since, for every linear objective function vector $c \in \mathbb{R}^n$, we have $\max\{\langle c, x\rangle : x \in \mathrm{P}^{\text{perm}}(n)\} = \max\{\sum_{i=1}^n \sum_{j=1}^n j c_i y_{ij} : y \in \mathrm{P}^{\text{birk}}(n)\}$, one can use $\mathrm{P}^{\text{birk}}(n)$ (that can be described by n^2 nonnegativity inequalities) instead of $\mathrm{P}^{\text{perm}}(n)$ (whose description requires $2^n - 2$ inequalities) with respect to linear programming related issues.

In general, an *extension* of a polytope $P \subseteq \mathbb{R}^n$ is a polyhedron $Q \subseteq \mathbb{R}^d$ (i.e., an intersection of finitely many affine hyperplanes and halfspaces) together with a linear projection $p : \mathbb{R}^d \to \mathbb{R}^n$ satisfying $P = p(Q)$. Any description of Q by linear equations and linear inequalities then (together with p) is an *extended formulation* of P. The *size* of the extended formulation is the number of inequalities in the description. Note that we neither account for the number of equations (we can get rid of them by eliminating variables) nor for the number of variables (we can ensure that there are not more variables than inequalities by projecting Q to the orthogonal complement of its *lineality space*, where the latter is the space of all directions of lines contained in Q). If $T \in \mathbb{R}^{n\times d}$ is the matrix with $p(y) = Ty$, then, for every $c \in \mathbb{R}^n$, we have $\max\{\langle c, x\rangle : x \in P\} = \max\{\langle T^t c, y\rangle : y \in Q\}$.

In the example described above, $\mathrm{P}^{\text{birk}}(n)$ thus provides an extended formulation of $\mathrm{P}^{\text{perm}}(n)$ of size n^2. It is not known whether one can do something similar for the matching polytopes $\mathrm{P}^{\text{match}}(n)$ (we will be back to this question in Section 4.2). However there are many other examples of nice and small extended formulations for polytopes associated with combinatorial optimization problems. The aim of this article is to show a few of them and to shed some light on the geometric, combinatorial and algebraic background of this concept that recently has received increased attention. The presentation is not meant to be a survey (for this purpose, we refer to Vanderbeck and Wolsey [31] as well as to Cornuéjols, Conforti, and Zambelli [11]) but rather an appetizer for investigating alternative possibilities to express combinatorial optimization problems by means of linear programs.

While we will not be concerned with practical aspects here, extended formulations have also proven to be useful in computations. You will find more on this in Laurence Wolsey's discussion column below. Fundamental work with respect to understanding the concept of extended formulations and its limits has been done by Mihalis Yannakakis in his 1991-paper *Expressing Combinatorial Optimization Problems by Linear Programs* [34] (see Sect. 3.3 and 4). We are very happy that he shares with us some of his thoughts on the subject in another discussion column.

2 Some Examples

2.1 Spanning Trees

The *spanning tree polytope* $\mathrm{P}^{\text{spt}}(n)$ associated with the complete graph $K_n = (V_n, E_n)$ on n nodes is the convex hull of all characteristic vectors of spanning trees, i.e., of all subsets of edges that form connected and cycle-free subgraphs. In another seminal paper, Edmonds [14] proved that $\mathrm{P}^{\text{spt}}(n)$ is the set of all $x \in \mathbb{R}^{E_n}_+$ that satisfy the equation $x(E_n) = n - 1$ and the inequalities $x(E_n(S)) \leq |S| - 1$ for all $S \subseteq V_n$ with $2 \leq |S| < n$. Again, none of the exponentially many inequalities is redundant.

However, by introducing additional variables $z_{v,w,u}$ for all ordered triples (v, w, u) of pairwise different nodes meant to encode whether the edge $\{v, w\}$ is contained in the tree and u is in the component of w when removing $\{v, w\}$ from the tree, it turns out that the system consisting of the equations $x_{\{v,w\}} - z_{v,w,u} - z_{w,v,u} = 0$ and $x_{\{v,w\}} + \sum_{u\in[n]\setminus\{v,w\}} z_{v,u,w} = 1$ (for all pairwise different $v, w, u \in V_n$) along with the nonnegativity constraints and the equation $x(E_n) = n - 1$ provides an extended formulation of $\mathrm{P}^{\text{spt}}(n)$ of size $O(n^3)$ (with orthogonal projection to the space of x-variables). This formulation is due to Martin [23] (see also [34, 11]). You will find an alternative one in Laurence Wolsey's discussion column below.

2.2 Disjunctive Programming

If $P_i \subseteq \mathbb{R}^n$ is a polytope for each $i \in [q]$, then clearly $P = \text{conv}(P_1 \cup \cdots \cup P_q)$ is a polytope as well, but, in general, it is difficult to derive a description by linear equations and inequalities in $\mathbb{R}^n$ from such descriptions of the polytopes P_i. However constructing an extended formulation for P in this situation is very simple. Indeed suppose that each P_i is described by a system $A^i x \leq b^i$ of f_i linear inequalities (where, in order to simplify notation, we assume that equations are written, e.g., as pairs of inequalities). Then the system $A^i z^i \le \lambda_i b^i$ for all $i \in [q]$, $\sum_{i=1}^{q} \lambda_i = 1$, $\lambda \ge \mathbb{O}$ with variables $z^i \in \mathbb{R}^n$ for all $i \in [q]$ and $\lambda \in \mathbb{R}^q$ is an extended formulation for P of size $f_1 + \cdots + f_q + q$, where the projection is given by $(z^1, \ldots, z^q, \lambda) \mapsto z^1 + \cdots + z^q$. This has been proved first by Balas (see, e.g., [3]), even for polyhedra that are not necessarily polytopes (where in this general case P needs to be defined as the topological closure of the convex hull of the union).

2.3 Dynamic Programming

When a combinatorial optimization problem can be solved by a dynamic programming algorithm, one often can derive an extended formulation for the associated polytope whose size is roughly bounded by the running time of the algorithm.

A simple example is the 0/1-Knapsack problem, where we are given a nonnegative integral weight vector $w \in \mathbb{N}^n$, a weight bound $W \in \mathbb{N}$, and a profit vector $c \in \mathbb{R}^n$, and the task is to solve $\max\{\langle c, x\rangle : x \in F(w, W)\}$ with $F(w, W) = \{x \in \{0, 1\}^n : \langle w, x\rangle \le W\}$. A classical dynamic programming algorithm works by setting up an acyclic directed graph with nodes $s = (0, 0)$, t, and (i, ω) for all $i \in [n]$, $\omega \in \{0, 1, \ldots, W\}$ and arcs from (i, ω) to (i', ω') if and only if $i < i'$ and $\omega' = \omega + w_{i'}$, where such an arc would be assigned length $c_{i'}$, as well as arcs from all nodes to t (of length zero). Then solving the 0/1-Knapsack problem is equivalent to finding a longest s-t-path in this acyclic directed network, which can be carried out in linear time in the number α of arcs.

The polyhedron $Q \subseteq \mathbb{R}^{\alpha}_{+}$ of all s-t-flows of value one in that network equals the convex hull of all characteristic vectors of s-t-paths (due to the total unimodularity of the node-arc incidence matrix), thus it is easily seen to be mapped to the associated *Knapsack-polytope* $\mathrm{P}^{\mathrm{knap}}(w, W) = \mathrm{conv}(F(w, W))$ via the projection given by $y \mapsto x$, where x_i is the sum of all components of y indexed by arcs pointing to nodes of type $(i, \star)$. As Q is described by nonnegativity constraints, the flow-conservation equations on the nodes different from s and t and the equation ensuring an outflow of value one from s, these constraints provide an extended formulation for $\mathrm{P}^{\mathrm{knap}}(w, W)$ of size α.

However quite often dynamic programming algorithms can only be formulated as longest-paths problems in acyclic directed *hypergraphs* with hyperarcs of the type (S, v) (with a subset S of nodes) whose usage in the path represents the fact that the optimal solution to the partial problem represented by node v has been constructed from the optimal solutions to the partial problems represented by the set S. Martin, Rardin, and Campbell [24] showed that, under the condition that one can assign appropriate *reference sets* to the nodes, also in this more general situation nonnegativity constraints and flow-equations suffice to describe the convex hull of the characteristic vectors of the hyperpaths. This generalization allows one to derive polynomial size extended formulations for many of the combinatorial optimization problems that can be solved in polynomial time by dynamic programming algorithms.

2.4 Others

A common generalization of the techniques to construct extended formulations by means of disjunctive programming or dynamic programming is provided by *branched polyhedral systems* (*BPS*) [20]. In this framework, one starts from an acyclic directed graph that has associated with each of its non-sink nodes v a polyhedron in the space indexed by the out-neighbors of v. From these building blocks, one constructs a polyhedron in the space indexed by all nodes. Under certain conditions one can derive an extended formulation for the constructed polyhedron from extended formulations of the polyhedra associated with the nodes.

Some very nice extended formulations have recently been given by Faenza, Oriolo, and Stauffer [16] for stable set polytopes of claw-free graphs. Here the crucial step is to glue together descriptions of stable set polytopes of certain building block graphs by means of *strip compositions*. One of their constructions can be obtained by applying the BPS-framework, though apparently the most interesting one they have cannot.

An asymptotically smallest possible extended formulation of size $O(n \log n)$ for the permutahedron $\mathrm{P}^{\mathrm{perm}}(n)$ has been found by Goemans [18]. His construction relies on the existence of *sorting networks* of size $r = O(n \log n)$ (Ajtai, Komlós, and Szemerédi [1]), i.e., sequences $(i_1, j_1), \ldots, (i_r, j_r)$ for which the algorithm that in each step s swaps elements a_{i_s} and a_{j_s} if and only if $a_{i_s} > a_{j_s}$ sorts every sequence $(a_1, \ldots, a_n) \in \mathbb{R}$ into non-decreasing order. The construction principle of Goemans has been generalized to the framework of *reflection relations* [21], which, for instance, can be used to obtain small extended formulations for all *G-permutahedra of finite reflection groups G* (see, e.g., Humphreys [19]), including extended formulations of size $O(\log m)$ of regular m-gons, previously constructed by Ben-Tal and Nemirovski [6]. Another application of reflection relations yields extended formulations of size $O(n \log n)$ for *Huffman-polytopes*, i.e., the convex hulls of the leaves-to-root-distances vectors in rooted binary trees with n labelled leaves. Note that linear descriptions of these polytopes in the original spaces are very large, rather complicated, and unknown (see Nguyen, Nguyen, and Maurras [26]).

The list of combinatorial problems for which small (and nice) extended formulations have been found comprises many others, among them perfect matching polytopes of planar graphs (Barahona [5]), perfectly matchable subgraph polytopes of bipartite graphs (Balas and Pulleyblank [4]), stable-set polytopes of distance claw-free graphs (Pulleyblank and Shepherd [28]), packing and partitioning orbitopes [15], subtour-elimination polytopes (Yannakakis [34] and, for planar graphs, Rivin [30], Cheung [9]), and certain mixed-integer programs (see, e.g., Conforti, di Summa, Eisenbrand, and Wolsey [12]).

3 Combinatorial, Geometric, and Algebraic Background

3.1 Face Lattices

Any intersection of a polyhedron P with the boundary hyperplane of some affine halfspace containing P is called a *face* of P. The empty set and P itself are considered to be (non-proper) faces of P as well. The proper faces of a three-dimensional polytope thus are its vertices, edges, and the polygons that make up the boundary of P. Partially ordered by inclusion, the faces of a polyhedron P form a lattice $\mathcal{L}(P)$, the *face lattice* of P. The proper faces that are maximal with respect to inclusion are the *facets* of P. Equivalently, the facets of P are those faces whose dimension is one less than the dimension of P. An irredundant linear description of P has exactly one inequality for each facet of P.

If $Q \subseteq \mathbb{R}^d$ is an extension of the polytope $P \subseteq \mathbb{R}^n$ with a linear projection $p : \mathbb{R}^d \to \mathbb{R}^n$, then mapping each face of P to its preimage in Q under p defines an embedding of $\mathcal{L}(P)$ into $\mathcal{L}(Q)$. Figure 1 illustrates this embedding for the trivial extension $Q = \{y \in \mathbb{R}^V_+ : \sum_{x \in X} y_x = 1\}$ of $P = \mathrm{conv}(X)$ via $p(y) = \sum_{x \in X} y_x x$ for $X = \{\mathbb{e}_1, -\mathbb{e}_1, \ldots, \mathbb{e}_4, -\mathbb{e}_4\}$ (thus P is the *cross-polytope* in $\mathbb{R}^4$ with 16 facets and Q is the *standard-simplex* in $\mathbb{R}^8$ with 8 facets). As this figure suggests, constructing a small extended formulation for a polytope P means to hide the facets of P in the fat middle part of the face lattice of an extension with few facets.

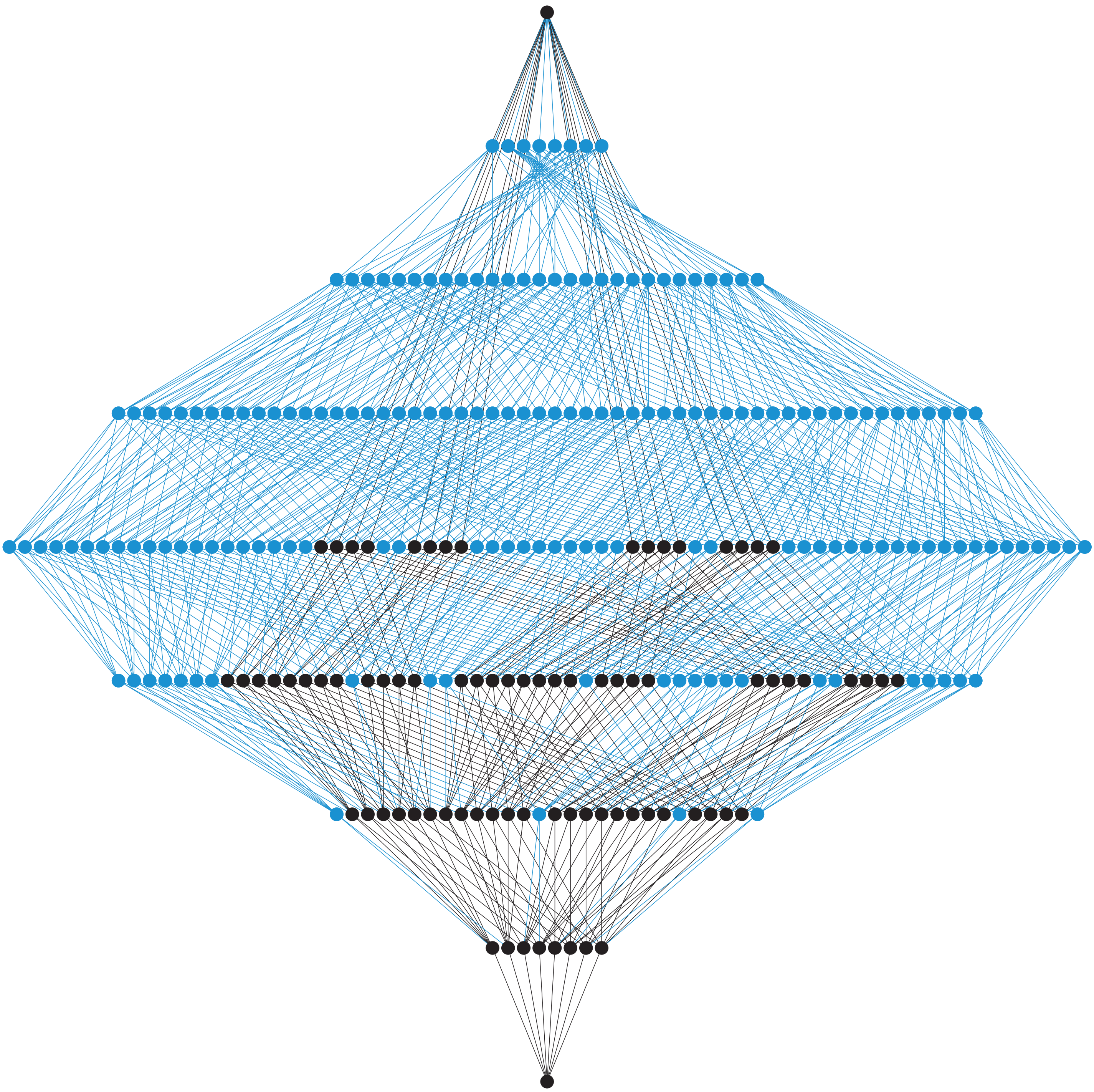

Figure 1. *Embedding of the face lattice of the* 4*-dimensional cross-polytope into the face lattice of the* 7*-dimensional simplex*

3.2 Slack Representations

Let $P = \{x \in \mathcal{A} : Ax \le b\} \subseteq \mathbb{R}^n$ be a polytope with affine hull $\mathcal{A} = \mathrm{aff}(P)$, $A \in \mathbb{R}^{m \times n}$, and $b \in \mathbb{R}^m$. The affine map $\varphi : \mathcal{A} \to \mathbb{R}^m$ with $\varphi(x) = b - Ax$ (the *slack map* of P w.r.t. $Ax \le b$) is injective. We denote its inverse (the *inverse slack map*) on its image, the affine subspace $\tilde{\mathcal{A}} = \varphi(\mathcal{A}) \subseteq \mathbb{R}^m$, by $\tilde{\varphi} : \tilde{\mathcal{A}} \to \mathcal{A}$. The polytope $\tilde{P} = \tilde{\mathcal{A}} \cap \mathbb{R}^m_+$, the *slack-representation* of P w.r.t. $Ax \le b$, is isomorphic to P with $\varphi(P) = \tilde{P}$ and $\tilde{\varphi}(\tilde{P}) = P$.

If $Z \subseteq \mathbb{R}^m_+$ is a finite set of nonnegative vectors whose *convex conic hull* $\mathrm{ccone}(Z) = \{\sum_{z \in Z} \lambda_z z : \lambda \ge \mathbb{O}\} \subseteq \mathbb{R}^m_+$ contains $\tilde{P} = \tilde{\mathcal{A}} \cap \mathbb{R}^m_+$, then we have $\tilde{P} = \tilde{\mathcal{A}} \cap \mathrm{ccone}(Z)$, and thus, the system $\sum_{z \in Z} \lambda_z z \in \tilde{\mathcal{A}}$ and $\lambda_z \ge 0$ (for all $z \in Z$) provides an extended formulation of P of size $|Z|$ via the projection $\lambda \mapsto \tilde{\varphi}(\sum_{z \in Z} \lambda_z z)$. Let us call such an extension a *slack extension* and the set Z a *slack generating set* of P (both w.r.t. $Ax \le b$).

Now suppose conversely that we have any extended formulation of P of size q defining an extension Q that is *pointed* (i.e., the polyhedron Q does not contain a line). As for polytopes above (which in particular are pointed polyhedra), we can consider a slack representation $\tilde{Q} \subseteq \mathbb{R}^q$ of Q and the corresponding inverse slack map $\tilde{\psi}$. Then we have $\varphi(p(\tilde{\psi}(\tilde{Q}))) = \tilde{P}$, where p is the projection map of the extension. If the system $Ax \le b$ is *binding* for P, i.e., each of its inequalities is satisfied at equation by some point from P, then one can show (by using strong LP-duality) that there is a *nonnegative* matrix $T \in \mathbb{R}^{m \times q}_+$ with $\varphi(p(\tilde{\psi}(\tilde{z}))) = T\tilde{z}$ for all $\tilde{z} \in \tilde{Q}$, thus $\tilde{P} = T\tilde{Q}$. Hence the columns of T form a slack generating set of P

(w.r.t. $Ax \le b$), yielding a slack extension of size q. As every non-pointed extension of a polytope can be turned into a pointed one of the same size by projection to the orthogonal complement of the lineality space, we obtain the following result, where the *extension complexity* of a polytope P is the smallest size of any extended formulation of P.

Theorem 1 ([17]). *The extension complexity of a polytope P is equal to the minimum size of all slack extensions of P.*

As every slack extension of a polytope is bounded (and since all bounded polyhedra are polytopes), Theorem 1 implies that the extension complexity of a polytope is attained by an extension that is a polytope itself. Furthermore, in Theorem 1 one may take the minimum over the slack extensions w.r.t. any fixed binding system of inequalities describing P. In particular, all these minima concide.

3.3 Nonnegative Rank

Now let $P = \mathrm{conv}(X) = \{x \in \mathrm{aff}(P) : Ax \le b\} \subseteq \mathbb{R}^n$ be a polytope with some finite set $X \subseteq \mathbb{R}^n$ and $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$. The *slack matrix* of P w.r.t. X and $Ax \le b$ is $\Phi \in \mathbb{R}_+^{[m]\times X}$ with $\Phi_{i,x} = b - \langle A_{i,\star}, x\rangle$. Thus the slack representation $\tilde{P} \subseteq \mathbb{R}^m$ of P (w.r.t. $Ax \le b$) is the convex hull of the columns of Φ. Consequently, if the columns of a nonnegative matrix $T \subseteq \mathbb{R}_+^{[m]\times[f]}$ form a slack generating set of P, then there is a nonnegative matrix $S \in \mathbb{R}_+^{[f]\times X}$ with $\Phi = TS$. Conversely, for every factorization $\Phi = T'S'$ of the slack matrix into nonnegative matrices $T' \in \mathbb{R}_+^{[m]\times[f']}$ and $S' \in \mathbb{R}_+^{[f']\times X}$, the columns of T' form a slack generating set for P.

Therefore constructing an extended formulation of size f for P amounts to finding a factorization of the slack matrix $\Phi = TS$ into nonnegative matrices T with f columns and S with f rows. In particular, we have derived the following result that essentially is due to Yannakakis [34] (see also [17]). Here, the *nonnegative rank* of a matrix is the minumum number f such that the matrix can be written as a product of two nonnegative matrices, where the first one has f columns and the second one has f rows.

Theorem 2. *The extension complexity of a polytope P is equal to the nonnegative rank of its slack matrix (w.r.t. any set X and binding system $Ax \le b$ with $P = \mathrm{conv}(X) = \{x \in \mathrm{aff}(P) : Ax \le b\}$).*

Clearly, the nonnegative rank of a matrix is bounded from below by its usual rank as known from Linear Algebra. There is also quite some interest in the *nonnegative* rank of (not necessarily slack) matrices in general (see, e.g., Cohen and Rothblum [10]).

4 Fundamental Limits

4.1 General Lower Bounds

Every extension Q of a polytope P has at least as many faces as P, as the face lattice of P can be embedded into the face lattice of Q (see Sect. 3.1). Since each face is the intersection of some facets, one finds that the extension complexity of a polyhedron with β faces is at least $\log\beta$. This observation has first been made by Goemans [18] in order to argue that the extension complexity of the permutahedron $\mathrm{P}^{\mathrm{perm}}(n)$ is at least $\Omega(n\log n)$.

Suppose that $\Phi = TS$ is a factorization of a slack matrix Φ of the polytope P into nonnegative matrices T and S with columns $t^1,\dots,t^f$ and rows $s^1,\dots,s^f$, respectively. Then we can write $\Phi = \sum_{i=1}^f t^i s^i$ as the sum of f nonnegative matrices of rank one. Calling the set of all non-zero positions of a matrix its *support*, we thus find that the nonnegative factorization $\Phi = TS$ provides a way to cover the support of Φ by f *rectangles*, i.e., sets of the form $I \times J$, where I and J are subsets of the row- and column-indices of Φ, respectively. Hence, due to Theorem 2, the minimum number of rectangles by which one can cover the support of Φ yields a lower bound (the *rectangle covering bound*) on the extension complexity of P (Yannakakis [34]). Actually, the rectangle covering bound dominates the bound discussed in the previous paragraph [17]. As Yannakakis [34] observed furthermore, the logarithm of the rectangle covering bound of a polytope P is equal to the *nondeterministic communication complexity* of the predicate on the pairs (v,f) of vertices v and facets f of P that is true if and only if $v \notin f$.

One can equivalently describe the rectangle covering bound as the minimum number of complete bipartite subgraphs needed to cover the *vertex-facet-non-incidence graph* of the polytope P. A *fooling set* is a subset F of the edges of this graph such that no two of the edges in F are contained in a complete bipartite subgraph. Thus every fooling set F proves that the rectangle covering bound, and hence, the extension complexity of P, is at least $|F|$. For instance, for the n-dimensional cube it is not too difficult to come up with a fooling set of size $2n$, proving that for a cube one cannot do better by allowing extended formulations for the representation. For more details on bounds of this type we refer to [17].

Unfortunately, all in all the currently known techniques for deriving lower bounds on extension complexities are rather limited and yield mostly quite unsatisfying bounds.

4.2 The Role of Symmetry

Asking, for instance, about the extension complexity of the matching polytope $\mathrm{P}^{\mathrm{match}}(n)$ defined in the beginning, one finds that not much is known. It might be anything between quadratic and exponential in n. However, in the main part of his striking paper [34], Yannakakis established an exponential lower bound on the sizes of *symmetric* extended formulations of $\mathrm{P}^{\mathrm{match}}(n)$. Here, *symmetric* means that the extension polyhedron remains unchanged when renumbering the nodes of the complete graph, or more formally that, for each permutation π of the edges of the complete graph that is induced by a permutation of its nodes, there is a permutation κ_π of the variables of the extended formulation that maps the extension polyhedron to itself such that, for every vector y in the extended space, applying π to the projection of y yields the same vector as projecting the vector obtained from y by applying κ_π. Indeed, many extended formulations are symmetric in a similar way, for instance the extended formulation of the permutahedron by the Birkhoff-polytope mentioned in the Introduction as well as the extended formulation for the spanning tree polytope discussed in Section 2.1.

In order to state Yannakakis' result more precisely, denote by $\mathcal{M}_\ell(n)$ the set of all matchings of cardinality ℓ in the complete graph with n nodes, and by $\mathrm{P}_\ell^{\mathrm{match}}(n) = \mathrm{conv}\{\chi(M) : M \in \mathcal{M}_\ell(n)\}$ the associated polytope. In particular, $\mathrm{P}_{n/2}^{\mathrm{match}}(n)$ is the *perfect-matching-polytope* (for even n).

Theorem 3 (Yannakakis [34]). *For even n, the size of every symmetric extended formulation of $\mathrm{P}_{n/2}^{\mathrm{match}}(n)$ is at least $\Omega(\binom{n}{\lfloor (n-2)/4\rfloor})$.*

Since $\mathrm{P}_{\lfloor n/2\rfloor}^{\mathrm{match}}(n)$ is (isomorphic to) a face of $\mathrm{P}^{\mathrm{match}}(n)$, one easily derives the above mentioned exponential lower bound on the sizes of symmetric extended formulations for $\mathrm{P}^{\mathrm{match}}(n)$ from Theorem 3.

At the core of his beautiful proof of Theorem 3, Yannakakis shows that, for even n, there is no symmetric extended formulation in equation form (i.e., with equations and nonnegativity constraints only) of $\mathrm{P}_{n/2}^{\mathrm{match}}(n)$ of size at most $\binom{n}{k}$ with $k = \lfloor (n-2)/4\rfloor$. From such a hypothetical extended formulation EF_1, he first constructs an extended formulation EF_2 in equation form on variables y_A for all matchings A with $|A| \le k$ such that the 0/1-vector valued map $s^\star$ on the vertices of $\mathrm{P}_{n/2}^{\mathrm{match}}(n)$ defined by $s^\star(\chi(M))_A = 1$ if and only if $A \subseteq M$ is a *section* of EF_2, i.e., $s^\star(x)$ maps every vertex x to a preimage under the projection of EF_2 that is contained in the extension polyhedron. Then it turns out that an extended formulation like EF_2 cannot exist. In fact, for an arbitrary partitioning of the node set into two parts V_1 and V_2 with $|V_1| = 2k+1$, one can construct a nonnegative point $y^\star$ in the affine hull of the image of $s^\star$ (thus $y^\star$ is contained in the extension polyhedron of EF_2 that is defined by equations and nonnegativity constraints only) with $y^\star_{\{e\}} = 0$ for all edges e connecting V_1 and V_2, which implies that the projection of the point $y^\star$ violates the inequality $x(\delta(V_1)) \geq 1$ that is valid for $\mathrm{P}^{\mathrm{match}}_{n/2}(n)$ (since $|V_1| = 2k+1$ is odd). The crucial ingredient for constructing EF_2 from EF_1 is a theorem of Bocherts' [8] stating that every subgroup G of permutations of m elements that is primitive with $|G| > m!/\lfloor(m+1)/2\rfloor!$ contains all even permutations. Yannakakis constructs a section s for EF_1 for that he can show – by exploiting Bochert's theorem – that there is a nonnegative matrix C with $s(\chi(M)) = C \cdot s^\star(\chi(M))$ for all $M \in \mathcal{M}_{n/2}(n)$, which makes it rather straight forward to construct EF_2 from EF_1.

With respect to the fact that his proof yields an exponential lower bound only for *symmetric* extended formulations, Yannakakis [34] remarked "we do not think that asymmetry helps much" in constructing small extended formulations of the (perfect) matching polytopes and stated as an open problem to "prove that the matching (...) polytopes cannot be expressed by polynomial size LP's without the symmetry assumption". As indicated above, today we still do not know whether this is possible. However, at least it turned out recently that requiring symmetry can make a big difference for the smallest possible size of an extended formulation.

Theorem 4 ([22]). *All symmetric extended formulations of $\mathrm{P}^{\mathrm{match}}_{\lfloor \log n\rfloor}(n)$ have size at least $n^{\Omega(\log n)}$, while there are polynomial size non-symmetric extended formulations for $\mathrm{P}^{\mathrm{match}}_{\lfloor \log n\rfloor}(n)$ (i.e., the extension complexity of $\mathrm{P}^{\mathrm{match}}_{\lfloor \log n\rfloor}(n)$ is bounded from above by a polynomial in n).*

Thus, at least when considering matchings of size $\lfloor \log n\rfloor$ instead of perfect (or arbitrary) matchings, asymmetry indeed helps much.

While the proof of the lower bound on the sizes of symmetric extended formulations stated in Theorem 4 is a modification of Yannakakis' proof indicated above, the construction of the polynomial size non-symmetric extended formulation of $\mathrm{P}^{\mathrm{match}}_{\lfloor \log n\rfloor}(n)$ relies on the principle of disjunctive programming (see Section 2.2). For an arbitrary coloring ζ of the n nodes of the complete graph with $2k$ colors, we call a matching M (with $|M| = k$) *ζ-colorful* if, in each of the $2k$ color classes, there is exactly one node that is an endnode of one of the edges from M. Let us denote by P_ζ the convex hull of the characteristic vectors of ζ-colorful matchings. The crucial observation is that P_ζ can be described by $O(2^k + n^2)$ inequalities (as opposed to $\Omega(2^n)$ inequalities needed to describe the polytope associated with all matchings, see the Introduction). On the other hand, according to a theorem due to Alon, Yuster, and Zwick [2], there is a family of q such colorings $\zeta_1, \ldots, \zeta_q$ with $q = 2^{O(k)} \log n$ such that, for every $2k$-element subset W of the n nodes, in at least one of the colorings the nodes from W receive pairwise different colors. Thus we have $\mathrm{P}^{\mathrm{match}}_k(n) = \mathrm{conv}(P_{\zeta_1} \cup \cdots \cup P_{\zeta_q})$, and hence (as described in Section 2.2) we obtain an extended formulation of $\mathrm{P}^{\mathrm{match}}_k(n)$ of size $2^{O(k)} n^2 \log n$, which, for $k = \lfloor \log n\rfloor$, yields the upper bound in Theorem 4.

Yannakakis [34] moreover deduced from Theorem 3 that there are no polynomial size symmetric extended formulations for the traveling salesman polytope (the convex hull of the characteristic vectors of all cycles of lengths n in the complete graph with n nodes). Similarly to Theorem 4, one can also prove that there are no polynomial size symmetric extended formulations for the polytopes associated with cycles of length $\lfloor \log n\rfloor$, while these polytopes nevertheless have polynomially bounded extension complexity [22].

Pashkovich [27] further extended Yannakakis' techniques in order to prove that every symmetric extended formulation of the permutahedron $\mathrm{P}^{\mathrm{perm}}(n)$ has size at least $\Omega(n^2)$, showing that the Birkhoff-polytope essentially provides an optimal *symmetric* extension for the permutahedron.

5 Conclusions

Many polytopes associated with combinatorial optimization problems can be represented in small, simple, and nice ways as projections of higher dimensional polyhedra. Moreover, though we have not touched this topic here, sometimes such extended formulations are also very helpful in deriving descriptions in the original spaces. What we currently lack are on the one hand more techniques to construct extended formulations and on the other hand a good understanding of the fundamental limits of such representations. For instance, does every polynomially solvable combinatorial optimization problem admit an extended formulation of polynomial size? We even do not know this for the matching problem. How about the stable set problem in perfect graphs? The best upper bound on the extension complexity of these polytopes for graphs with n nodes still is $n^{O(\log n)}$ (Yannakakis [34]).

Progress on such questions will eventually shed more light onto the principle possiblities to express combinatorial problems by means of linear constraints. Moreover, the search for extended formulations yields new modelling ideas some of which may prove to be useful also in practical contexts. In any case, work on extended formulations can lead into fascinating mathematics.

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Discussion Column

Laurence A. Wolsey

Using Extended Formulations in Practice

Though Dantzig-Wolfe decomposition [7] dating from 1960 and the description of the convex hull of the union of polyhedra of Balas [2] from the early 1970s can both be seen as results about using additional variables in modeling, systematic interest in what we now call extended formulations (EFs) seems to have begun in the 1980s. The flurry of activity in polyhedral combinatorics in the 1970s concentrated on the development of valid inequalities to strengthen the formulation of both easy and NP-hard integer programs, and only later was the question of adding variables to obtain tighter formulations raised systematically.

In this short discussion, we try to indicate a few of the areas in which EFs have been used computationally, as well as the different ways in which they have been tested. Two recent surveys [5] and [22] contain many more examples of EFs and the techniques available for constructing such formulations, the first concentrating more on combinatorial optimization problems and the second more on integer programming. Note that we do not discuss the use of an EF to generate valid inequalities in the original space of variables, as in the case of Benders' algorithm [4] or lift-and-project [3].

Lot-sizing, network design and routing models are three of the areas in which EFs have been effective computationally. Below we will first describe one or two EFs in each of these areas. Then we will briefly indicate different ways in which they have been used.

Lot-Sizing – Direct Use of an MIP Solver

Lot-sizing problems provide perhaps the richest class of problems for which EFs have been developed that are small in size and computationally effective. Two reasons for this are the fact that the single item problem can be viewed as a fixed charge network flow problem and that the uncapacitated and constant capacity variants typically can be solved in polynomial time via dynamic programming. In 1977 Krarup and Bilde [13] published a reformulation of the single-item uncapacitated lot-sizing problem as an uncapacitated facility location problem and showed that for the specific objective function obtained for lot-sizing, the linear programming relaxation had integer solutions. Another way to explain this reformulation is as a multicommodity reformulation of the initial fixed charge network flow formulation in which the demand for each time period is treated as a separate commodity. Recently such a multicommodity reformulation has been shown to be effective computationally [15] for 2-level production-transportation problems with on the order of 2–5 production sites, 5–10 items, 10–40 clients and 12–24 time periods.

In an important paper in 1987 Eppen and Martin [8] showed that a shortest path EF of the same single item uncapacitated problem was actually an integer polytope. More generally they showed how in many cases a dynamic programming algorithm for optimization over a discrete set X leads through LP duality to an extended formulation whose projection onto the original space of variables gives the convex hull of solutions $\mathrm{conv}(X)$. They developed EFs for several different lot-sizing variants and among others solved to near-optimality multi-item big bucket lot-sizing problems with up to 200 products and 10 time periods.

Pochet and Wolsey [17] and Constantino [6] developed compact and tight EFs for several variants of the constant capacity single-item lot-sizing problem. Based on this work, Günlük and Pochet [12] later showed that the important structure underlying many of these results was a simple mixed integer set of the form $\{(s,x) \in \mathbb{R} \times \mathbb{Z}^n : s + x_t \geq b_t,\ t = 1,\ldots,n\}$, called the mixing set. The book [18] describes an automatic reformulation procedure to enable the user to benefit from these EFs and contains several production planning case studies using them. The direct approach used for the computational results cited so far is to take the multi-item problem, add the EFs for each item, and then feed the resulting formulation directly to an MIP solver.

The Tree Polytope and Routing – Using Approximate EFs

A well-known formulation for the set X of incidence vectors of spanning trees in a graph consists of degree constraints and an exponential number of subtour elimination constraints using variables $x \in \mathbb{R}^{|E|}$ where $x_e = 1$ denoting that edge e is in the spanning tree. Wong [23] proposed an EF for the symmetric traveling salesman problem that provides a tight formulation for the spanning tree polytope. Specifically to model the connectivity of a spanning tree, one chooses a root node r and then constructs an arborescence rooted at r in which it is possible to direct a unit of flow from the root to each node $i \neq r$. This leads to a formulation as a single source fixed charge network flow problem on the graph (V,E): $\{(x,y,f) \in \{0,1\}^{|E|} \times \{0,1\}^{|A|} \times \mathbb{R}_+^{|A|} : x_e = y_{ij} + y_{ji}\ e = (i,j) \in E,\ \sum_j f_{ij} - \sum_j f_{ji} = -1\ i \neq r,\ f_{ij} \leq (|V|-1)y_{ij}\ (i,j) \in A\}$, where $y_{ij} = 1$ if arc (i,j) is in the arborescence and f_{ij} is the flow in arc (i,j). Again introducing a distinct commodity for each node $k \neq r$, one obtains the EF $\{(x,y,w) \in \{0,1\}^{|E|} \times \{0,1\}^{|A|} \times \mathbb{R}_+^{|A|(|V|-1)} : x_e = y_{ij} + y_{ji}\ \forall e = (i,j) \in E,\ \sum_j w_{ij}^k - \sum_j w_{ij}^k = 0\ \forall k, i \neq r, k,\ \sum_j w_{jk}^k = 1\ \forall k \neq r,\ w_{ij}^k \leq y_{ij} \forall (i,j) \in A,\ k\}$ where $w_{ij}^k = 1$ if the directed path from the root to node k passes through the arc (i,j). With the cardinality constraint $\sum_{e \in E} x_e = |V| - 1$, this EF is an alternative to that of Martin for the convex hull of incidence vectors of spanning trees described above by Kaibel. This formulation involves $O(n^3)$ variables and constraints if $n = |V|$ which leads to formulations that are too large for practical use if n exceeds 30-40. In [21], see also [1], one relaxes or approximates this formulation to get one of more reasonable size. Specifically the idea is to drop the k,i flow conservation constraint if node i is not within some selected neighborhood of node k, as well as the variables w_{ij}^k, w_{ji}^k if neither i or j is in the neighborhood. This relaxation allows one to solve traveling salesman problems with 70 or so nodes directly with an MIP solver.

Multi-commodity Fixed Charge Network Flows – Column and Row Generation

Magnanti et al. [14] considered the following model for a single arc: $X = \{(x,y) \in \mathbb{R}_+^K \times \mathbb{Z}_+^1 : \sum_{k=1}^K d_k x_k \leq Cy, x \leq 1\}$, and showed that the exponential family of Residual Capacity or MIR inequalities: $\sum_{j \in T} d_j x_j \leq \alpha_T + \beta_T y$ where $T \subseteq \{1,\ldots,K\}$, $\rho_T = \lfloor \frac{d(T)}{C} \rfloor$, $\beta_T = d(T) - \rho C$ and $\alpha_T = (C - \beta_T)\rho_T$ provide the convex hull. Recently Frangioni and Gendron [9] have shown that the following polyhedron Q provides an EF for $\text{conv}(X)$: $\{(x,y,v,w) \in \mathbb{R}_+^K \times \mathbb{R}_+^1 \times \mathbb{R}_+^S \times \mathbb{R}_+^{KS} : \sum_{s=1}^S v_s \leq 1,\ (s-1)Cv_s \leq \sum_k d_k w_{ks} \leq sCv_s\ \forall\ s,\ w_{ks} \leq v_s\ \forall\ k,s,\ y = \sum_{s=1}^S v_s, x_k = \sum_{s=1}^S w_{ks}\ \forall\ k\}$, where $S = \lceil \frac{\sum_{k=1}^K d_k}{C} \rceil$ and we can interpret the additional variables as follows: $v_s = 1$ if $y = s$ and $v_s = 0$ otherwise, and $w_{ks} = x_k$ if $y = s$ and $w_{ks} = 0$ otherwise. Here the formulation obtained by adding this EF for each arc is too large to be solved directly by an MIP solver. The authors develop a column and row generation algorithm which works with a subset of the variables and constraints, solves an optimization problem over each single arc set to find missing variables that need to be added, and then simultaneously adds the constraints in which these variables occur. The authors use the approach to obtain strong lower bounds for multicommodity capacitated network design problems with up to 30 nodes and 400 commodities.

Parallel Machine Scheduling – Cutting Planes

Given a set of n jobs, a natural first choice is the set of variables: t_j denoting the start time of job j. If the processing times of the jobs are integer, the time-indexed variables [19] where $w_{jt} = 1$ if job j starts processing in time t is a common choice for additional variables. These allow one to model many machine scheduling problems, but already the resulting extended formulations are too large to be fed directly to an MIP solver. However surprisingly it can be of interest to consider an even larger set of variables, such as $z_{ijt} = 1$ if job i finishes and job j starts at time t on the same machine, and job 0 corresponds to idle. With these variables, it is not difficult to see that the following holds for any subset $S \subseteq N$ of jobs: $\sum_{t=1}^T \sum_{i \in S, j \notin S} tz_{ijt} - \sum_{t=1}^T \sum_{i \notin S, j \in S} tz_{ijt} = \sum_{j \in S} p_j$, where T is an upper bound on the time horizon. Setting $u_t = \sum_{i \in S, j \notin S} z_{ijt}$ and $v_t = \sum_{i \notin S, j \in S} z_{ijt}$, this becomes the knapsack set $\{(u,v) \in \mathbb{Z}_+^T \times \mathbb{Z}_+^T : \sum_{t=1}^T tu_t - \sum_{t=1}^T tv_t = \sum_{j \in S} p_j\}$, for which one can generate cutting planes. Note that these cannot in general be converted into cutting planes in either the t_j variables or in the w_{jt} variables. In a recent paper of Pessoa et al. [16] such cutting planes form one of the important computational steps of an algorithm allowing the solution of parallel machine scheduling problems with up to 4 machines and 100 jobs. This idea appeared in a paper of Gouveia [11], and has been used in several successful studies in vehicle routing, [10], [20], etc. combining cutting planes in both the original and additional variables and column generation, under the title branch-and-price-and-cut.

We have not discussed in detail the size of the EFs presented above. However if they are more than quadratic in the size of the original formulation, it is typically not possible to solve them directly with an MIP solver. In such cases considerable algorithmic originality is required to successfully use EFs and this is an intriguing area for research. Apart from the approaches discussed her, two more standard directions include the use of EFs to develop heuristics, see for instance [8], and to derive valid inequalities in the original space (variants of Benders' algorithm).

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Mihalis Yannakakis

On Extended LP Formulations

In connection with his paper, Volker Kaibel asked me to give some background on my paper "Expressing Combinatorial Optimization Problems by Linear Programs" [3] regarding the motivation and thoughts that guided that work. That research was carried out in 1987 and was presented first at the STOC'88 conference. It was motivated on the one hand by a claimed proof of P=NP that appeared at that time and attracted a lot of attention in the community, and on the other hand by the developments in the preceding years in Linear Programming and the polyhedral approach to combinatorial optimization.

In 1986–87 E. R. Swart circulated a paper that claimed to solve the Traveling Salesman Problem using Linear Programming [2]. In particular, the paper constructed a large LP with many extra variables (n^8 in the original version, n^{10} in a revision), which purportedly provided an extended formulation of the TSP polytope. Given the size and complexity of the LP, it was rather hard to determine what was exactly the effect of these variables and constraints and whether the LP worked or not. It was clear that some methodology was needed to understand what is possible to achieve with extra variables and whether this approach could possibly work in principle.

The approach itself is actually a reasonable one to try if one believes that P=NP. First, Linear Programming is a P-complete problem, so it is in some sense a hardest, universal problem in P. Second, we know that the introduction of extra variables (which are then existentially quantified, i.e. projected out) is a powerful tool in various domains that can increase drastically the expressive power of a model, turning an exponential object into a polynomial one. There are for example simple polytopes with an exponential number of facets, that can be expressed succinctly with extra variables. In logic, every Boolean predicate can be represented by an equivalent Boolean formula (in conjunctive normal form), but the formula requires often exponential size. However, introducing extra variables we can express any NP predicate by a polynomial-size formula; this is essentially Cook's theorem showing that Satisfiability is NP-complete.

Another motivation came from the discovery of Khachian's ellipsoid algorithm and its applications, and Karmakar's algorithm. Grötchel, Lovasz and Schrijver had shown that the ellipsoid algorithm could be used to solve in polynomial time various problems (such as the clique and independent set problem on perfect graphs) even though their standard LP formulation has an exponential number of constraints (provided there is a good separation algorithm for the constraints). In view of the impracticality of the ellipsoid algorithm, it would be desirable to use instead Karmakar's algorithm (or Simplex) for these problems; however this would require a polynomial size LP description, which raises the question whether one can construct such a formulation using any (small) set of extra variables and constraints. The same question is relevant also for NP-hard problems, such as the TSP, with respect to useful classes of their facets; presumably (if P≠NP) we cannot construct exact extended formulations in polynomial time, but perhaps we can express succinctly important classes of facets that provide useful cutting planes.

With this background I tried to take a systematic look to get some understanding of what can and cannot be achieved with compact extended LP formulations. The paper resolved partially some questions and left many more open. For general polytopes a pleasant surprise was that the complexity of the smallest extended LP formulation can be characterized by a concrete parameter, the nonnegative rank of the slack matrix. The problem is that it is often not that easy to compute the nonnegative rank. The paper pointed out a connection to communication complexity, which can be used sometimes to obtain lower or upper bounds, and applied it to the independent (stable) set polytope for some classes of graphs. In general however, we still need to develop good methods to estimate or bound the nonnegative rank.

With respect to the question of expressing the TSP polytope by a small LP, I could not show that this is impossible, but could rule it out at least for the class of symmetric LPs (which includes the proposed LP in Swart's attempted proof): any such formulation must have exponential size. It was easier to prove this result first for the (perfect) matching polytope and then transfer it by reduction to the TSP polytope. (The result holds for a somewhat larger class of LPs but it was simpler to state it for symmetric LPs.) In subsequent years there have been several more similar attempts to prove P=NP using an extended LP formulation for the TSP; they are generally symmetric or almost so (sometimes for example a particular node is singled out as the starting node of the tour; the exponential lower bound holds even if a constant fraction of nodes is singled out.) Symmetry seems to be rather natural in the construction of exact LP formulations, but as shown recently by Kaibel, Pashkovich and Theis [1], nonsymmetry can save a superpolynomial amount in some cases.

It remains an intriguing open question whether the matching polytope can be expressed by a polynomial-size (unrestricted) LP. For the

TSP and the polytopes of other NP-hard problems, we expect that this must be impossible. Is this tantamount to showing P≠NP? It does not seem so. The P=NP question is equivalent to a related but somewhat different question, reflecting in a sense the difference between decision and optimization problems: P=NP iff we can construct efficiently a polynomial-size extended LP formulation of a polyhedron that includes the characteristic vectors of Hamiltonian graphs and excludes those of non-Hamiltonian. Such an LP could be easily obtained from a compact LP formulation for the TSP polytope, but the converse does not seem to hold in any obvious way. I believe in fact that it should be possible to prove that there is no polynomial-size formulation for the TSP polytope or any other NP-hard problem, although of course showing this remains a challenging task.

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Announcements

Mathematical Programming, Series A, is going online

Starting in January, 2011 all submissions to *Mathematical Programming, Series A*, should be made through Springer's *Online Manuscript Submission, Review and Tracking System* for the journal, at the web site www.editorialmanager.com/mapr/.

Most scholarly journals published by major publishing companies have such online submission systems, and the move to such a system was long overdue for *Mathematical Programming*. There are many advantages of such a system – starting with the submission, the editorial progress of each paper is recorded, the author can check on the status of his/her submission, automatic reminders are sent to the editors if they are late, etc. Moreover, statistics on the workload and performance of each editor are easy to obtain. One important detail is that the online system, at least initially, will be for the use of authors and the editorial board only – all communications between the editors and referees will remain on a personal level, outside of the system.

We hope that this move will improve the overall performance of the journal and will help to eliminate outlier cases where an author waits in frustration for a long-overdue report on his/her paper. Of course, as with any new system some tuning will probably be necessary to deal with unforseen problems, but the initial testing phase went smoothly. Once the online system is well established for *Series A* of *Mathematical Programming*, it may also be adapted for the use of *Series B* of the journal.

We believe that this change is an important step in the journal's organization, and are confident that it will result in a more reliable and professional service to the optimization community.

Kurt Anstreicher (MPA Editor-in-Chief)
Alexander Shapiro (Chair of the MOS Publications Committee)
Philippe Toint (Chair of the MOS)

Call for nomination for the 2012 George B. Dantzig Prize

Nominations are solicited for the George B. Dantzig Prize, administered jointly by the Mathematical Optimization Society (MOS) and the Society for Industrial and Applied Mathematics (SIAM). This prize is awarded to one or more individuals for original research which by its originality, breadth and depth, is having a major impact on the field of mathematical optimization. The contribution(s) for which the award is made must be publicly available and may belong to any aspect of mathematical optimization in its broadest sense.

The prize will be presented at the *2012 International Symposium on Mathematical Programming*, to be held August 19–24, 2012, in Berlin, Germany. The members of the prize committee are John Birge (Chair), Gerard Cornuejols, Yuri Nesterov, and Eva Tardos.

Nominations should consist of a letter describing the nominee's qualifications for the prize, and a current curriculum vitae of the nominee including a list of publications. They should be sent to

John Birge
University of Chicago
Booth School of Business
5807 South Woodlawn Avenue
Chicago, IL 60637, USA
Email: John.Birge@ChicagoBooth.edu

and received by 15 November 2011. Submission of nomination materials in electronic form is strongly encouraged.

Mixed Integer Programming 2011

June 20–23, 2011, University of Waterloo, Canada

You are cordially invited to participate in the upcoming workshop in Mixed Integer Programming (MIP 2011). The 2011 Mixed Integer Programming workshop will be the eighth in a series of annual workshops held in North America designed to bring the integer programming community together to discuss very recent developments in the field. The workshop is especially focused on providing opportunities for junior researchers to present their most recent work. The workshop series consists of a single track of invited talks. MIP 2011 is scheduled immediately following IPCO XV, which will take place at IBM T.J. Watson Research Center in Yorktown Heights, NY from June 15–17 (http://ipco2011.uai.cl).

Confirmed speakers ◦ Amitabh Basu – UC Davis ◦ Gerard Cornuejols – Carnegie Mellon University ◦ Claudia D'Ambrosio – University of Bologna ◦ Santanu Dey – Georgia Tech ◦ Sarah Drewes – UC Berkeley ◦ Samir Elhedhli – University of Waterloo ◦ Marcos Goycoolea – Universidad Adolfo Ibanez ◦ Willem-Jan van Hoeve – Carnegie Mellon University ◦ Adam Letchford – Lancaster University ◦ Leo Liberti – Ecole Polytechnique ◦ Marco Luebbecke – RWTH Aachen University ◦ Susan Margulies – Rice University ◦ Alex Martin – Universitt Erlangen-Nrnberg ◦ Giacomo Nannicini – Carnegie Mellon University ◦ Michael Perregaard – FICO ◦ Sebastian Pokutta – MIT ◦ Oleg Prokopyev – University of Pittsburgh ◦ Sebastian Sager – University of Heidelberg ◦ Domenico Salvagnin – University of Padova ◦ Gautier Stauffer – University of Bordeaux I ◦ Laura Sanita – Ecole Polythechnique Federale de Lausanne ◦ Levent Tuncel – University of Waterloo ◦ Francois Vanderbeck – University of Bordeaux I ◦ Robert Weismantel – ETH Zurich

The workshop is designed to provide ample time for discussion and interaction between the participants, as one of its aims is to facilitate research collaboration. Thanks to the generous support by our sponsors, registration is free, and travel support is available.

Program Committee: Shabbir Ahmed (Georgia Institute of Technology), Ricardo Fukasawa (University of Waterloo), Ted Ralphs (Lehigh University), Juan Pablo Vielma (University of Pittsburgh), Giacomo Zambelli (London School of Economics).

www.math.uwaterloo.ca/~mip2011/

Optimization 2011

July 24–27, 2011, Lisbon (Caparica), Portugal, Department of Mathematics, School of Sciences and Technology, New University of Lisbon

Optimization 2011 is the seventh edition of a series of Optimization international conferences held every three or four years, in Portugal. This meeting strives to bring together researchers and practitioners from different areas and with distinct backgrounds, but with common interests in optimization. This conference series has international recognition as an important forum of discussion and exchange of ideas, being organized under the auspices of APDIO (the Portuguese Operations Research Society).

In this edition, we feel honored to celebrate the 60th anniversary of our dear colleague Joaquim Joao Judice (Univ. of Coimbra).

Confirmed plenary speakers: Gilbert Laporte (HEC Montreal), Jean Bernard Lasserre (LAAS-CNRS, Toulouse), Jose Mario Martinez (State University of Campinas), Mauricio G.C. Resende (AT&T Labs – Research), Nick Sahinidis (Carnegie Mellon University), Stephen J. Wright (University of Wisconsin).

We look forward to meeting you in *Optimization 2011*.
Ana Luisa Custodio (Co-chair of the Organizing Committee)
Paula Amaral (Co-chair of the Organizing Committee)

http://www.fct.unl.pt/optimization2011

MOPTA 2011

August 17–19, 2011, Lehigh University, Rauch Business Center, Bethlehem, PA, USA

MOPTA aims at bringing together a diverse group of people from both discrete and continuous optimization, working on both theoretical and applied aspects. There will be a small number of invited talks from distinguished speakers and contributed talks, spread over three days.

Our target is to present a diverse set of exciting new developments from different optimization areas while at the same time providing a setting which will allow increased interaction among the participants. We strive to bring together researchers from both the theoretical and applied communities who do not usually have the chance to interact in the framework of a medium-scale event.

Confirmed plenary speakers: Mark Daskin (U. of Michigan), Michael Ferris (U. of Wisconsin), Adrian Lewis (Cornell U), Jorge More (Argonne), Javier Pena (Carnegie Mellon), Cliff Stein (Columbia U), Philippe Toint (U of Namur).

Organizing Committee: Katya Scheinberg (Chair), Tamás Terlaky, Ted Ralphs, Robert Storer, Aurélie Thiele, Larry Snyder, Frank E. Curtis.

We look forward to seeing you at MOPTA 2011.

http://coral.ie.lehigh.edu/~mopta/

OR 2011

August 30 – September 2, 2011, Zurich, Switzerland

The main goal of the conference is to bring together members of the international OR community to discuss scientific progresses in various subfields of OR in a truly interdisciplinary spirit. The highlights and core of the conference are the invited Keynote Speakers and the parallel semi-plenary lectures on various topics representing the state of the art in these fields. Certainly, the conference provides a platform to present current research and to compete for a publication in the referred proceedings.

Plenary lectures: Dimitris J. Bertsimas (MIT, Cambridge): *Advances in stochastic and adaptive optimization*; Kenneth L. Judd (Hoover Institution, Stanford): *Numerically Efficient and Stable Algorithms for Solving Large Dynamic Programming Problems in Economics, Finance, and Climate Change Models*; William Pulleyblank (United States Military Academy, West Point, NY): *Challenges and Opportunities for Operations Research in the next decade.*

Program committee: Karl Schmedders (Chair, University of Zurich), Friedrich Eisenbrand (EPF Lausanne), Luca Gambardella (IDSIA, Lugano), Diethard Klatte (University of Zurich), Ulrike Leopold-Wildburger (University of Graz), Hans-Jakob Lüthi (ETH Zurich), Stefan Nickel (Karlsruhe Institute of Technology), Stefan Pickl (Universität der Bundeswehr München), Marion Rauner (University of Vienna), Brigitte Werners (Ruhr-Universität Bochum)

Organization: The Swiss Association of Operations Research (SVOR) is the premier organization in Switzerland for advancing the profession, practice, and science of operations research (OR) and management science (MS). Every four years the german speaking OR societies from Austria (ÖGOR), Germany (GOR) and Switzerland (SVOR) organize a joint international conference *OR 2011*.

The local organizers are the Institute for Operations Research (IFOR, Prof. H.-J. Lüthi) of the ETH Zurich and the Institute for OR (IOR, Prof. K. Schmedders) of the University of Zurich who – under the patronage of SVOR – will share the organizational, financial and scientific responsibilities.

http://www.or2011.ch

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IPCO 2011
The 15th Conference on Integer Programming and Combinatorial Optimization

IPCO XV will be held on June 15–17, 2011 at the IBM T. J. Watson Research Center in Yorktown Heights, New York, USA.

Accepted papers (in order of submission) ◦ Amitabh Basu, Gerard Cornuejols and Marco Molinaro. A probabilistic analysis of the strength of the split and triangle closures ◦ Alexander Ageev, Yohann Benchetrit, Andras Sebo and Zoltan Szigeti. An Excluded Minor Characterization of Seymour Graphs ◦ Stephan Held, Edward C. Sewell and William Cook. Safe Lower Bounds For Graph Coloring ◦ Mathieu Van Vyve. Fixed-charge transportation on a path: Linear programming formulations ◦ Britta Peis and Andreas Wiese. Universal packet routing with arbitrary bandwidths and transit times ◦ Santanu S. Dey and Sebastian Pokutta. Design and Verify: A New Scheme for Generating Cutting-Planes ◦ Jose A. Soto and Claudio Telha. Jump Number of Two-Directional Orthogonal Ray Graphs ◦ Anna Karlin, Claire Mathieu and Thach Nguyen. Integrality Gaps of Linear and Semi-definite Programming Relaxations for Knapsack ◦ Daniel Dadush, Santanu S. Dey and Juan Pablo Vielma. On the Chvatal-Gomory Closure of a Compact Convex Set ◦ Deeparnab Chakrabarty, Chandra Chekuri, Sanjeev Khanna and Nitish Korula. Approximability of Capacitated Network Design ◦ Deeparnab Chakrabarty and Chaitanya Swamy. Facility Location with Client Latencies ◦ Satoru Iwata and Mizuyo Takamatsu. Computing the Maximum Degree of Minors in Mixed Polynomial Matrices via Combinatorial Relaxation ◦ Pierre Bonami. Lift-and-Project Cuts for Mixed Integer Convex Programs ◦ Fabrizio Grandoni and Thomas Rothvoss. Approximation Algorithms for Single and Multi-Commodity Connected Facility Location ◦ Kenjiro Takazawa. Discrete convexity and faster algorithms for weighted matching forests ◦ Tamas Kiraly and Lap Chi Lau. Degree Bounded Forest Covering ◦ Monia Giandomenico, Adam Letchford, Fabrizio Rossi and Stefano Smriglio. A New Approach to the Stable Set Problem Based on Ellipsoids ◦ Aman Dhesi, Pranav Gupta, Amit Kumar, Gyana Parija and Sambuddha Roy. Contact Center Scheduling with Strict Resource Requirements ◦ Volker Kaibel and Kanstantsin Pashkovich. Constructing Extended Formulations from Reflection Relations ◦ Sylvia Boyd, Rene Sitters, Suzanne van der Ster and Leen Stougie. TSP on Cubic and Subcubic Graphs ◦ Yu Hin Au and Levent Tuncel. Complexity Analyses of Bienstock-Zuckerberg and Lasserre Relaxations on the Matching and Stable Set Polytopes ◦ Mario Ruthmair and Günther Raidl. A Layered Graph Model and an Adaptive Layers Framework to Solve Delay-Constrained Minimum Tree Problems ◦ Inge Gortz, Marco Molinaro, Viswanath Nagarajan and R Ravi. Capacitated Vehicle Routing with Non-Uniform Speeds ◦ S. Thomas McCormick and Britta Peis. A primal-dual algorithm for weighted abstract cut packing ◦ Friedrich Eisenbrand, Naonori Kakimura, Thomas Rothvoss and Laura Sanita. Set Covering with Ordered Replacement – Additive and Multiplicative Gaps ◦ Oliver Friedmann. A subexponential lower bound for Zadeh's pivoting rule for solving linear programs and games ◦ Claudia D'Ambrosio, Jeff Linderoth and James Luedtke. Valid Inequalities for the Pooling Problem with Binary Variables ◦ Bissan Ghaddar, Juan Vera and Miguel Anjos. An Iterative Scheme for Valid Polynomial Inequality Generation in Binary Polynomial Programming ◦ Martin Bergner, Alberto Caprara, Fabio Furini, Marco Lübbecke, Enrico Malaguti and Emiliano Traversi. Partial Convexification of General MIPs by Dantzig-Wolfe Reformulation ◦ Trang Nguyen, Mohit Tawarmalani and Jean-Philippe Richard. Convexification Techniques for Linear Complementarity Constraints ◦ William Cook, Thorsten Koch, Daniel Steffy and Kati Wolter. An exact rational mixed-integer programming solver ◦ Ojas Parekh. Iterative packing for demand matching and sparse packing ◦ Matteo Fischetti and Michele Monaci. Backdoor branching

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