OPTIMA 87

Mathematical Optimization Society Newsletter

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MOS Chair's Column

November 1, 2011. Here is yet another issue of Optima packed with goodies central to our field. After the summer months most of us are now back to our more usual occupations and our research activities in optimization. I truly hope that you share my anticipation of its moments of collaborative inspiration. One thing is sure: after the successful mid-year meetings, we are now heading towards the high point of 2012: the ISMP in Berlin. I hear from good sources that preparations are progressing well, and that all augurs are favourable.

As you all know, several prizes will be awarded at the ISMP opening ceremony, recognizing the contributions or both younger and more senior colleagues. You undoubtedly have seen the various calls for nominations for the Dantzig, Lagrange, Fulkerson, Beale-Orchard-Hays and Tucker prizes as well as that for the Paul Tseng lectureship. I encourage you to seriously consider nominating one or more optimization researchers for these prizes. These awards and the high scientific standards of their recipients not only recognize the talents in our field, but also significantly improve its visibility to scientists in different domains and to the general public. Even if not essential, this remains useful in our times of media frenzy ...

I could not conclude this column without congratulating Richard Tapia for his recent nomination by president Obama for the US National Medal of Science. I have known Richard for many years and, besides his insightful technical achievements, I have always admired his determination to provide students and researchers from Mexico and elsewhere with the most valuable guidance and opportunities – an effort that often revealed talented people to the international community. That he is recognized today for both this achievements is only justice. Congratulations, dear Richard, in the name of us all.

Finally, I just heard from the MOS office that the emails have been sent to encourage you to renew your membership in the Society. Please take this suggestion seriously and renew or become a member: your affiliation does make a difference in the worldwide standing of the Society, and this is in our common interest.

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Note from the Editors

Whether or not there exists a polynomial-time variant of the simplex method depends on the geometry of polyhedra, and Hirsch conjectured in 1957 that, for any polyhedron defined by n inequalities in d dimensions, the length of the longest shortest path between any two vertices is at most n-d. When Francisco Santos announced his counterexample to the Hirsch conjecture in spring 2010, it sparked a flurry of activity examining the geometry and complexity of linear programming.

Jesus De Loera writes our main article in this issue. He paints a vivid landscape of these new developments in linear optimization, particularly from a geometric viewpoint, and even goes inside the polyhedron to survey the geometry of interior-point methods. Guenter Ziegler also provides his perspective, challenging us to push farther and faster into this area. Both authors see an exciting and fruitful future for linear programming, a topic which is so fundamental to our field but still harbors many mysteries.

Sam Burer, Co-Editor
Volker Kaibel, Co-Editor
Katya Scheinberg, Editor

Jesús A. De Loera

New Insights into the Complexity and Geometry of Linear Optimization

Linear programming is a pillar for computation and theory in mathematical optimization. For example, optimization problems with discrete variables are often reduced via branching to repeated use of linear programming [45]. Linear programs are also used in various approximation schemes for combinatorial and non-linear optimization (see e.g., [52, 100]) and in exciting new applications (e.g., [19, 20]). But the impact of linear optimization goes well beyond optimization and reaches other areas of mathematical research, e.g., in combinatorics and graph theory [53, 90]; and more recently the solution of the Kepler's conjecture required sophisticated linear programming techniques [55]. There are several excellent books ([14, 74, 91, 96]) and surveys on the theory and complexity of linear optimization (such as [79, 95]) covering advances up to 2002.

This article recounts recent exciting progress in the theory of linear optimization. I do not try to cover all advances, which would be impossible, but instead I focus on the geometric ideas that arose in recent times. In fact 2010 was an *annus mirabilis* for the theory of linear programming and most of the results I review here were presented at the workshop "*Efficiency of the Simplex Method: Quo vadis Hirsch conjecture?*" which took place January 18–21, 2011 at the Institute for Pure and Applied Mathematics at the University of California

Los Angeles (IPAM). I hope this article hints at the beautiful geometry that provide us with new opportunity. Just like Günter M. Ziegler suggested in [105], geometry does contribute to understand linear programs! So, while we assume the reader is somewhat familiar with linear programming, we will visit some of its finer geometric details.

In what follows I assume the feasibility region of a linear program is a rational convex polyhedron $P \subset \mathbb{R}^d$ given by the set of solutions of a the system of the form $Ax \leq b$, where A is an integral matrix with d-dimensional row vectors $a_1, \ldots, a_n$. Thus $P = \{x \in \mathbb{R}^d : Ax \leq b\}$, where $A \in \mathbb{Z}^{n \times d}$. For the most part we assume that vectors $a_1, \ldots, a_n$ span $\mathbb{R}^d$. In this way, the *input size* is given by d variables, n constraints, and the maximum binary size L of any coefficient of the data. We remark that in later sections, depending on the computations being discussed, we may change to other input format or presentations. When a polyhedron is bounded, we call it a *polytope* and they will be the main object of this article. A linear program is called *non-degenerate* if there is no $x \in P$ that satisfies $d + 1$ or more of the defining inequalities as equations. The corresponding polyhedron is called *simple*. It is well known that in discussing the complexity of linear programming there is no loss in generality in restricting the discussion to simple polyhedra, and we will mostly do this in this paper. The geometry of polytopes is clearly presented in the books [10, 104].

1 Advances related to the Simplex Method

Dantzig's simplex method from 1947 [24] and its variations are some of the most common algorithms for solving linear programs. It can be viewed as a family of combinatorial local search algorithms on the graph of a convex polyhedron. More precisely, the search is done over a finite graph, the one-skeleton of the polyhedron or *graph of the polyhedron*, which is composed of the zero and one dimensional faces of the feasible region (called *vertices* and *edges*). The search moves from a vertex of the one-skeleton to a better neighboring one according to some *pivot rule* which selects an improving neighbor. The operation of moving from one vertex to the next is called a *pivot step* or simply a *pivot*. Geometrically the simplex method traces a path on the graph of the polytope. Today, after sixty years of use and despite competition from interior point methods, the simplex method is still widely popular. The simplex method has even been selected as one of the most influential algorithms in the 20th century [32], but we still do not completely understand its theoretical performance.

One famous question is whether a polynomial-time version of the simplex method is possible. Such an algorithm would allow the solution of a linear program with a number of pivot steps that is a polynomial in d, n, and L. Is there even a strongly polynomial version, where the polynomial bound only depends on n and d? There are two big gaps of knowledge for uncovering the mystery: First, despite great effort of analysis, it remains open whether there is always a polynomial bound on the shortest path between two vertices in the skeleton. The *diameter* of the graph of a polytope is the length of the longest shortest path among all possible pairs of vertices. But even if we knew today a polynomial bound on the diameter of polyhedra, there is a second missing puzzle piece to decide the polynomiality of the simplex method. Klee and Minty first showed in 1972 [70] that pivot rules could be tricked into visiting 2^d vertices to find a path between two nodes, but the vertices are only one step apart in the skeleton of the cube. Since that achievement, most other pivot rules, including many favored in practical calculations, have been proved to be theoretically inefficient (see [5] and references therein). In the first section of this survey I summarize progress made on tackling these formidable obstacles.

Figure 1. *Francisco Santos (Photo by Komei Fukuda)*

1.1 *Francisco Santos' counterexample to the Hirsch conjecture*

Based on experiments Warren Hirsch conjectured in 1957 that the diameter of the graph of a polyhedron defined by n inequalities in d dimensions is at most $n - d$. Dantzig later popularized the conjecture when he published it in his well-known book [25]. Finally, after fifty three years of work by many researchers, *the Hirsch conjecture* has finally been disproved by a clever, elegant construction due to Francisco Santos of the University of Cantabria, Spain.

Santos first announced this to the world on May 10, 2010 when submitting a title and abstract for a talk at the conference "The Mathematics of Klee and Grünbaum: 100 years in Seattle". Hours later Gil Kalai posted the news on his popular blog [58]. The blog continues to attract activity around the subject because Gil Kalai has proposed a polymath project to attempt to finally find a polynomial bound for the diameter (more on this later!).

In what follows we will describe the key points of Santos' construction. In his initial announcement he showed the existence of a 43-dimensional polyhedron with 86 facets and diameter at least 44 (which as we see later has now been simplified). Before we begin, we wish to note that many researchers contributed, and continue to contribute, to this topic, so the interested reader should consult the survey [64] for full knowledge of what was known about the Hirsch conjecture up until early 2010.

A key observation of Santos' construction was an extension of a well-known result of Klee and Walkup. They proved in [68] that the Hirsch conjecture could be proved from just the case when $n = 2d$. In that case the problem is to prove that given two vertices u and v, that have no facet in common, one can pivot from one to the other in d steps so that at each pivot we abandon a facet containing u and enter a facet containing v. This was named the *d-step conjecture* (see also [64, 69]). The construction of Santos' counter-example requires a very clever variation of this result for a family of polytopes called *spindles*. Spindles are polytopes with two distinguished vertices u,v such that every facet contains either u or v but not both. Examples of a spindle include the cross polytopes and the polytope in Figure 2. Intuitively spindles can be seen as the

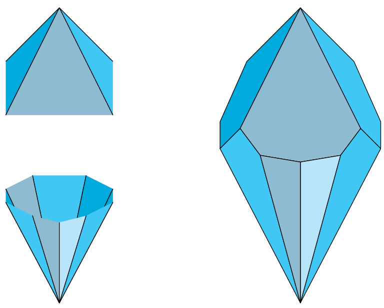

Figure 2. *A 3-spindle (picture courtesy of F. Santos)*

overlap of two pointed cones (as shown in the figure). The *length of a spindle* is the distance between this special pair of vertices. It is an entertaining exercise to prove that all 3-dimensional spindles have length three. Please try it out!

Santos's strong d-step theorem for spindles says that from a spindle P of dimension d, with $n > 2d$ facets and length λ one can construct another spindle P' of dimension $d+1$, with $n+1$ facets and length $\lambda+1$. Since one can repeat this construction again and again, each time increasing the dimension, length and number of facets of the new spindle by one unit we can repeat this process until we have $n = 2d$ (number of facets is twice the dimension). In particular, if a spindle P of dimension d with n facets has length greater than d then there is another spindle P' of dimension $n-d$, with $2n-2d$ facets, and length greater than $n-d$ which violates the Hirsch conjecture.

Santos wrote the bulk of the proof using the dual language of *prismatoids*, the polar duals of spindles. Under polarity, the vertices turn into facets and vice versa. A prismatoid is thus a polytope Q with two distinguished (parallel) facets F_1 and F_2 so that every vertex of Q is contained in exactly one of F_1 or F_2 (see top left of Figure 3). In terms of prismatoids, what used to be the length of the spindle, is now measured by the distance between the facets F_1, F_2 where one can move from one facet to another if they share a common face of dimension one less than $dim(F_i)$ Note, this is not the usual linear programming graph, we walk from face to face (as long as they are adjacent) rather than from vertex to vertex, and to make the distinction Santos called the facet distance from F_1 to F_2 its *width*. The width of the prismatoid is the same as the length of its dual spindle.

The objective is to construct a sequence of prismatoids that will eventually violate the Hirsch conjecture (remember this time on the facet distance, which is the normal graph distance in the spindle). The fundamental building block for the construction is the so called one-point suspension or, when we see it for spindles, the dual *wedge operation* (see Figure 4). Both operations were used before for constructions of polytopes satisfying the Hirsch bound with equality [56].

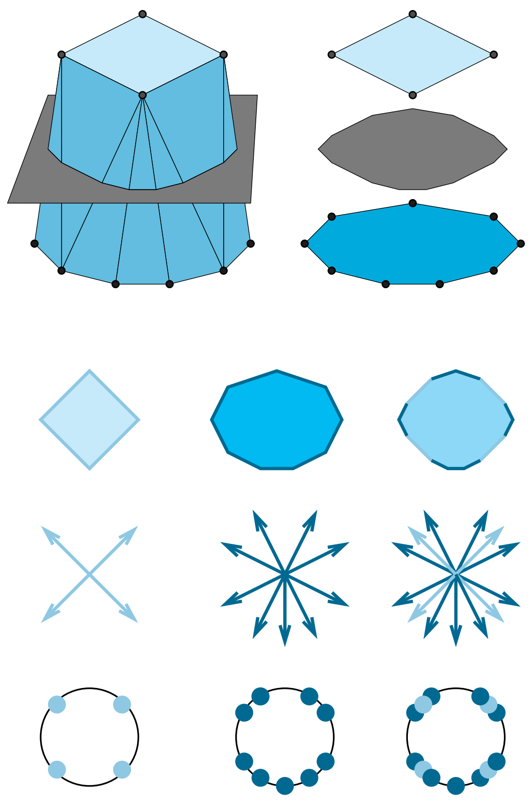

Figure 3. *(pictures courtesy of F. Santos)*

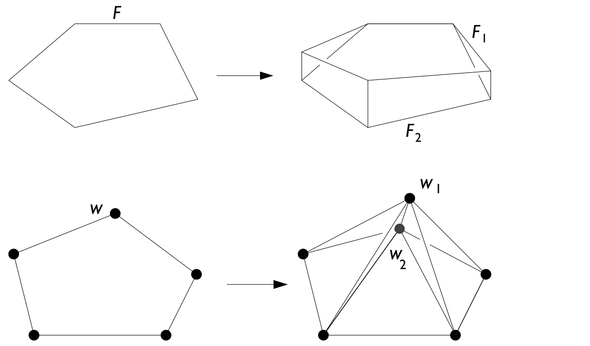

Figure 4. *A wedge construction and its dual the one point suspension (courtesy of Edward D. Kim)*

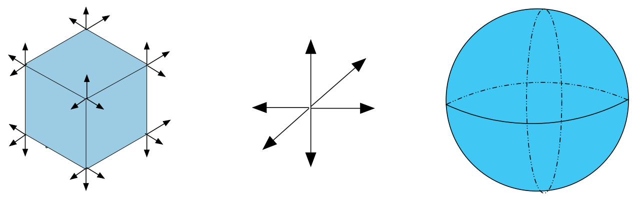

Figure 5. *The normal fan of a regular 3-cube has eight cones (left), which coincide with the eight possible orthants (center) To visualize and study the fan, we intersect it with the unit sphere. This yields a decomposition of S^2 into eight spherical triangles (right). The resulting map on the surface of the sphere is a combinatorial octahedron.*

Let v be a vertex of the polyhedron P, the *normal cone* C_v of v is the set of all vectors $c \in \mathbb{R}^n$ such that v is an optimal solution of the linear program $\max\{c^T x : x \in \mathbb{R}^n, Ax \leq b\}$. For example, the normal cone at a vertex of a regular cube is an orthant. The normal cone C_v of a vertex v is a full-dimensional polyhedral cone. Two vertices v and v' are adjacent if and only if C_v and $C_{v'}$ share a common facet. if P is a polytope, then the union of the normal cones of vertices of P is the complete space $\mathbb{R}^n$. The set union of all the normal cones for all vertices of the polytope defines the *normal fan*. See Figure 5 for an example of these notions.

Prismatoids are easier to analyze and visualize in a smaller dimension as the *Minkowski sum* of the distinguished facets F_1 and F_2 (see middle of Figure 3). The combinatorics of the Minkowski sum of F_1 and F_2 in turn can be seen from the superposition of the normal fans of F_1 and F_2 (see Figure 3 third level). This is also called the common refinement of the fans. For visualization and ease of analysis, the normal fans are often intersected with the unit sphere centered at the origin. This gives a *map* on the surface of the sphere, i.e., a decomposition of the sphere in question into spherical polyhedra. Santos' insight led him to see prismatoids as special superimposed maps over a sphere of even smaller dimension (see last level of Figure 3). In this way the normal fan of a $(d-1)$-polytope can be thought of as a map on the $(d-2)$-sphere. To disprove the Hirsch Conjecture Santos needed to find a prismatoid of dimension d and width larger than d. This task becomes finding a special pair of maps in the $(d-2)$ dimensional sphere that overlap in such a way that the resulting one-skeleton has a large diameter. More precisely, given two maps G_1 and G_2 in the $(d-2)$-sphere, that correspond to the prismatoid-defining facets F_1,F_2, the width of the prismatoid equals two plus the minimum number of steps needed to go from a vertex of G_1 to a vertex of G_2 in the (graph of the) superposition of the two maps.

Santos used some well-known properties of the 3-dimensional sphere S^3 to construct such a pair of maps. It has been known by topologists that S^3 can be obtained by symmetrically glueing two solid tori along their surface via the Hopf fibration. Each of the desired maps will be defined along parallel directions of one of the tori (a torus has two fundamental directions). This allows a decomposition of a three-dimensional sphere into the right kind of map resulting in a five-dimensional prismatoid of width six. Santos gave explicit coordinates for his prismatoid which can be found in his great article [87]. The polytope is so small that all of its properties can be checked "by hand". Finally, the polar dual of this nice prismatoid is the spindle necessary to obtain a counterexample to the Hirsch conjecture. Applying Santos' strong d-step theorem for spindles several times constructs a Hirsch counterexample that can be explicitly recovered from his five-prismatoid. More precisely he proved (see all missing details in the papers [87, 88])

Theorem 1. – *There is a* 43*-dimensional polytope with* 86 *facets and of diameter at least* 44*.*

– *There is an infinite family of non-Hirsch polytopes with* n *facets and diameter* $\sim (1+\epsilon)n$ *for some* $\epsilon > 0$*. This holds true even in fixed dimension.*

Since the first announcement and the release of Santos' solution there have been further developments: In [89] the authors proved that there are no prismatoids in four dimensions that would yield Hirsch counterexamples. Santos also showed that, using spindles (equivalently prismatoids) of fixed dimension, one cannot hope to get a polytope of superlinear diameter. More precisely, with spindles of dimension five, one cannot possibly surpass the Hirsch bound by more than fifty percent. Moreover currently the best value of ϵ in the theorem is $1/20$. Matschke, Santos and Weibel [77] have recently simplified Santos' original counterexample. The new initial prismatoid has only 25 vertices so that we now know the Hirsch conjecture fails in dimension 20, instead of the initial 43 dimensions (as the number of one-point suspensions is limited). This is significant as one can write the concrete inequalities defining a polyhedron with diameter larger than $n - d$. If you want to "touch" the non-Hirsch polytope, the inequalities are available at http://www.cs.dartmouth.edu/~weibel/hirsch.php?page=3.

Is there an upper bound for the diameter of polytopes which is polynomial in the number of facets and the dimension? The "Hirsch-bound barrier" has been finally broken after half a century of effort, but today we do not know the answer to this important question. The best general bound today, due to Kalai and Kleitman [61], is $O(n^{1+\log d})$, but in principle it could still be possible that there is always a linear diameter for polyhedra. In the next section we discuss recent efforts to procure general diameter bounds, both upper and lower bounds, for arbitrary polytopes.

1.2 Bounds for the diameter of polyhedra and a Polymath project

It has been pointed out repeatedly that the proofs of known upper bounds use only very limited properties of polytopes and often hold for more abstract complexes. For example, Klee and Kleinschmidt [69] showed that some bounds for fixed number of variables hold for the ridge-graphs of all pure simplicial complexes and more general objects. The *combinatorial-topological* approach to the diameter problem has a long history (see e.g., [72]); Adler, Dantzig, and Murty [1, 2] and Kalai [60] and others abstracted the notion of the graph of a polytope. This has become an important direction of research (see [6, 42, 46, 48, 65, 76, 80] and the many references therein). The objects one wishes to abstract are graphs of simple polyhedra. A d-dimensional polyhedron P is *simple* if each of its vertices is contained in exactly d of the n facets of P. It is well-known that it is enough to consider simple polyhedra, since the largest diameter of d-polyhedra with n facets must be achieved by a simple d-polyhedra with n facets.

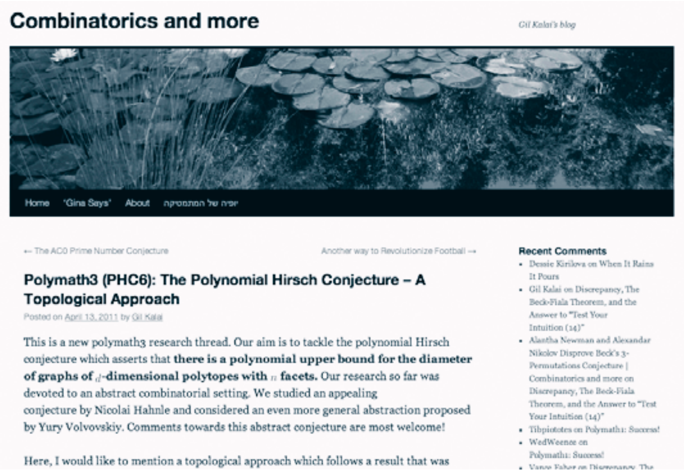

Figure 6. *A snapshot of Gil Kalai's blog and the Polymath3 project*

A recent exciting paper of Eisenbrand, Hähnle, Razborov, and Rothvoss [36] revisited combinatorial abstractions of polytope graphs by defining the properties of a *base abstraction* graph. Suppose $[n] = \{1, \ldots, n\}$ is an index set (intuition dictates to think of $[n]$ as the labels for the n facets of a polyhedron). Denote by $\binom{[n]}{d}$ the set of all d-element subsets of $[n]$. Let $\mathcal{A} \subseteq \binom{[n]}{d}$, and E is a set of unordered pairs of $\mathcal{A}$. If the graph $G = (\mathcal{A}, E)$ with the vertex set $\mathcal{A}$ and edge set E satisfies that (1) the graph G is connected, and (2) for each $A, A' \in \mathcal{A}$, there is a path from A to A' in the graph G using only vertices that contain $A \cap A'$, then we say that G is a d-dimensional *base abstraction* of $\mathcal{A}$ on the set $[n]$. The *diameter* of the base abstraction is the diameter of the graph G. It should be noted that the graphs of simple d-dimensional polyhedra with n facets are base abstractions because each of the n facets of P is associated with a label s in $[n]$. Since our polyhedron P is simple, each vertex of P is incident to exactly d facets, so it is associated with the d-element subset of $[n]$ consisting of the corresponding labels of facets. The graph G used in the base abstraction is the graph of the polyhedron. The first condition is satisfied since the graph of a polyhedron is connected (in fact d-connected, by Balinski's theorem). The second condition translates into the fact that for every pair of vertices y and z on a polyhedron P, there is a path from y to z on the smallest face of P containing both y and z.

A key idea of Eisenbrand et al. is that the diameter of a base abstraction graph can be derived from another combinatorial object, a *connected layer family*. A d-dimensional connected layer family on the set $[n]$ is a collection $\{F_0, \ldots, F_\delta\}$ of non-empty sets such that the elements of F_i are d-subsets of $[n]$ if $i \neq j$, then $F_i \cap F_j = \emptyset$. If $i < j < k$ and $u \in F_i$ and $w \in F_k$, then there is a $v \in F_j$ such that $u \cap w \subseteq v$ The connected layer family has diameter δ. From a connected layer family one can derive a base abstraction graph, the elements of each layer F_i are its vertices and all pairs of vertices u, v are connected when $u \in F_i$ and $v \in F_{i+1}$. The diameter of this graph would be δ. This translation in a more set-theoretic framework allowed them to use elegant tools from extremal combinatorics. Indeed, a great novelty in the Eisenbrand et al.'s approach is that there are base abstraction graphs with diameter greater than $\Omega(n^2/\log n)$, for a certain constant c. The construction makes elegant use of Lovász's local lemma applied to families of subsets forming covering designs.

Eisenbrand et al. were also able to prove that the base abstraction is a reasonable generalization because it satisfies some of the known upper bounds on the diameter of polytopes. Larman proved in [71] that for a d-dimensional polytope P with n facets the diameter is no more than $2^{d-3}n$ (shortly after this bound was improved by Barnette [9]). This bound shows that in fixed dimension the diameter must be linear in the number of facets. The best general bound of $O(n^{1+\log d})$ was obtained by Kalai and Kleitman [61]. The authors of [36] proved that the Larman bound and the Kalai-Kleitman bounds hold again for their graphs.

In response to the survey [64], the paper [36] and later Francisco Santos' solution of the Hirsch conjecture Gil Kalai initiated a wonderful internet-meta-collaboration. The polymath 3 project was started by Kalai and it lives within his blog [58]. We can read in http://polymathprojects.org/ that *"Polymath projects are massively collaborative mathematical research programs, in which a single problem, group of problems, or other mathematical task is worked on by a large group of mathematicians."* See [58] and Figure 6.

The online discussions have lead to many interesting and unexpected developments. For example, in his presentation at the January 2011 IPAM meeting, Nicolai Hähnle discussed a family of intermediate abstractions between the one in [36] (in which superlinear lower bounds and quasi-exponential upper bounds are known) and the abstraction of 1-reductions proposed by Volvosky in polymath3 (for which the quasi-polynomial upper bounds match, modulo a constant in the exponent, the lower bounds found). Hähnle also mentioned his multi-set generalization of the connected layer family model, in which it is very easy to construct graphs of diameter $d(n-1)$ and he showed that this upper bounds two extremal cases: when each layer consists of a single set, and when any of the possible $\binom{n+d-1}{d}$ sets is used in some family (it is worth remarking that an upper bound of $d(n-d)$ applies to the set version in the extremal case of using all sets; but the corresponding upper bound in the multi-set version is $d(n-1)$). He made a tantalizing conjecture:

Conjecture 2. *The diameter of a connected layer family and thus that of a polytope cannot exceed* $(d(n-1))$.

We already mentioned that the clever construction of two intersecting maps on the unit sphere S^3 is what allowed the construction of the five-prismatoid later used in the counterexample of the Hirsch conjecture. The online discussion taking place in the polymath and the work by F. Santos, T. Stephen and H. Thomas [89] motivated another recent fascinating conjecture of Gil Kalai related to pairs of maps in S^n that, if true, implies a polynomial bound on the diameter:

Conjecture 3. *Let M_1 be a red map and let M_2 be a blue map drawn in general position on S^n, and let M be their common refinement. Then there are vertices w of M, u a red vertex of M_1, v a blue vertex of M_2 and two faces F, G of M such that (1) $u, w \in F$, (2) $v, w \in G$, and (3) $dim(F) + dim(G) = n$.*

In Santos' construction, the red and blue maps come from red and blue polyhedral normal fans associated to red and blue convex polytope and the common refinement will be the fan obtained by taking all intersections of cones, one from the first fan and one from the second. An very recent paper (from September 2011) by Bonifas, Di Summa, Eisenbrand, Hähnle, and Niemeier [17] gives even more evidence that thinking in terms of normal fans of polytopes is a very good idea for bounding the diameter a polyhedron P. The bound they provided is polynomial in the dimension d and the largest absolute value of a sub-determinant of the defining integer matrix A which they denoted by Δ. More precisely,

Theorem 4 (Bonifas et al.). *Let $P = \{x \in \mathbb{R}^n : Ax \le b\}$ be a polytope where all sub-determinants of $A \in \mathbb{Z}^{n\times d}$ are bounded by Δ in absolute value. The diameter of the polyhedron P is bounded by $O(\Delta^2 d^4 \log(d\Delta))$. If P is bounded, then the diameter of P is at most $O(\Delta^2 d^{3.5} \log(d\Delta))$.*

Just like in Santos' construction of good prismatoids, the normal fan of the polytope plays a useful role in this bound. We consider the normal cones of sets of vertices intersected with the *unit ball* $B_d = \{x \in \mathbb{R}^d : \|x\|_2 \le 1\}$. Note that because the polytope is assumed to be simple, its normal cones are *simplicial* and thus the intersection of each normal cone is a simplicial spherical simplex. Traversing the graph of the original polytope translates into moving from one spherical simplex to the next as long as they share a facet. The main trick in the analysis is to reason about the volumes covered by the normal cones of vertices of P inside the unit ball. As before, denote by C_v the normal cone of the vertex v. For a set U of vertices of the polytope the volume considered is $\mathrm{vol}(U) := \mathrm{vol}\left(\bigcup_{v\in U} C_v \cap B_d\right)$. Given two vertices u, v of the polytope, to estimate how far apart they are we can consider visiting their neighbors, and then the neighbors of their neighbors, etc.(breadth-first search manner) until finally we find a common vertex. They used the volume of the cones involved to bound how many iterations can occur before this encounter. Clearly by the time the volume of the two sets of vertices (i.e., the union of their solid spherical cones) is more than half the volume of the unit ball they must have intersected. Thus the following geometric lemma yields the bounds on the diameter:

Lemma 5 (Bonifas et al.). *Let $P = \{x \in \mathbb{R}^d : Ax \le b\}$ be a polytope where all sub-determinants of $A \in \mathbb{Z}^{n\times d}$ are bounded by Δ in absolute value. Let $I \subseteq V$ be a set of vertices of the graph of the polytope with $\mathrm{vol}(I) \le (1/2) \cdot \mathrm{vol}(B_d)$. Then the volume of the neighborhood $\mathcal{N}(I)$ of I is at least*

$$\mathrm{vol}(\mathcal{N}(I)) \ge \sqrt{\frac{2}{\pi}} \frac{1}{\Delta^2 d^{2.5}} \cdot \mathrm{vol}(I).$$

Note that for the special case in which A is a totally unimodular matrix, the bounds for [17] simplify to $O(d^4 \log d)$ and $O(d^{3.5} \log d)$ respectively. This improves over the previous best bound for totally unimodular matrices by Dyer and Frieze [35]. This result suggests one should more easily obtain better bounds for the diameter of specific structured polytopes. Indeed, some new families of linear programs such as classical transportation and multiway transportation problems have been shown to satisfy linear and quadratic bounds respectively [18, 26].

To conclude we should remark that in a linear program the pivoting path will not be arbitrary, but the objective function dictates the pivots are monotonically increasing or decreasing, i.e., the vertices visited are ordered by the value of the objective function. A natural extremal question is then how long can monotone paths be in terms of n,d (and possibly L)? For more on this interesting topic see [105] and references therein.

1.3 Pivot rules and their bad behavior

A simplex method is governed by a pivoting rule, i.e., a method of choosing adjacent vertices with better objective function value. Starting with the historical 1972 construction of Klee and Minty [70], showing that Dantzig's original pivoting rule may require exponentially many steps, researchers debunked many of the popular pivot rules as good candidates for polynomial behavior. By 2010 almost all known natural deterministic pivoting rules were known to require an exponential number of steps to solve some linear programs (see [5, 105]), but three conspicuous pivot rules resisted the attacks of researchers until then. The most famous "untamed" pivot rules were Zadeh's rule (also known as the *least entered* rule) [102], the randomized pivot rules of Random-Edge originally proposed by G. Dantzig, and *Random-Facet* proposed by Kalai [59] and in a different form by Matousek, Sharir and Welzl [75].

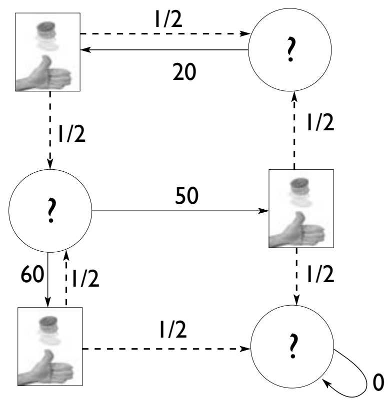

Figure 7. *An example of an MDP graph encoding the process*

At any non-optimal vertex, the Zadeh pivot rule chooses the decreasing edge that leaves the facet that has been left least often in the previous moves. In case of ties a tie-breaking rule is used to determine the decreasing edge to be taken. Any other pivot rule can be used as a tie-breaking rule. The rule was proposed by Norman Zadeh in a 1980 technical report from the department of operations research of Stanford University. It has now appeared reprinted in [102]. It was known for some time that Zadeh had offered $1000 for solving this problem (see e.g., [105]). Now, the random edge pivot rule chooses, from among all improving pivoting steps (or edges) from the current basic feasible solution (or vertex), one uniformly at random. The description of *Random-Facet* pivoting is a bit more complicated as there are several versions: roughly, at any non-optimal vertex v choose one facet F containing v uniformly at random and solve the problem restricted to F by applying the algorithm recursively to one of its facets. The recursion decreases the dimension of the polytope at each iteration, thus it will eventually restrict to a one-dimensional face, which is solved by following that edge. One repeats the process until reaching an optimum.

No non-polynomial lower bounds were known until recently for these three pivot rules. Prior evidence of exponential behavior was given in [73, 76] that both the random edge and random facet pivot rules do not have a polynomial bound when used in a certain class of oriented graphs which include the graph of a polyhedron oriented by an objective function. Morris [82] showed bad behavior existed for random edge in the related setting of linear complementarity problems (see also [6, 47] for more on abstract graphs). But there was also evidence of good behavior in special cases (e.g., [7, 57]) and the random facet rule can be shown to perform an expected subexponential number of steps in the worst case [59, 75]. This outperforms the deterministic pivot rules so far. So the result that these pivot rules are not always polynomial for specific simple LPs is an exciting breakthrough in the theory of the simplex method presented in the exciting papers [40, 41] by Oliver Friedmann, Thomas Dueholm Hansen, and Uri Zwick. See Figure 8.

Their new constructions use the close relation between simplex-type algorithms for solving linear programs and policy iteration algorithms for the stochastic 1-player games called *Markov decision processes*. Markov decision processes (MDP) model sequential decision-making in situations where outcomes are partly random and partly under the control of a decision maker (see [13, 84]). At each time step, the process is in some state i, and the decision maker chooses an action $j \in A_i$ that is available for state i. The process responds by randomly moving into a new state i', and giving the decision maker a corresponding reward $r_j(i,i')$. The probability that the process enters i as its new state is influenced by the chosen state-action. Specifically, it is given by the state transition function $P_j(i,i')$. Thus, the next state i' depends on the current state i and the decision makerÕs action j, but given i and j, it is conditionally independent of all previous states and actions. In all MDPs considered by Friedmann et al., one of the states is considered to be the initial state of the MDP, and a different state is considered to be the terminal state, or the sink, of the MDP. Also, the sink is reached with probability one after a finite number of steps, no matter which actions are chosen by the controller. This is needed to avoid infinite rewards.

A policy for the decision maker is a set function $p = p_1, p_2, \ldots, p_m$ that specifies the state-action p_i that the decision maker will choose when in state i. A policy is *positional* if it is deterministic and history independent. A policy is optimal if it maximizes the total expected reward. The goal of the decision maker is to maximize the total expected reward of all actions taken until reaching the sink by choosing a positional policy. One of the fundamental results concerning MDPs (see [84]) says that every MDP has an optimal positional policy. A key point is that one can obtain an LP that models the MDP problem in such a way that there is a one-to-one correspondence between policies of the MDP and basic feasible solutions of the LP, and between improving switches and improving pivots. This one-to-one correspondence only exists when you reach the terminal state with probability one from all states. An MDP model is conveniently represented as a bipartite graph that captures combinatorial information about the process (see Figure 7 for an example). Some nodes are the decision nodes that have arrows going to the randomization nodes (flipping coins) with a reward $r_j(i,i')$ as label. The other kind of arrow, going from randomization nodes to decision nodes, are labeled with the probabilities $P_j(i,i')$.

The breakthrough came in two nice papers. The team of Oliver Friedmann, Thomas Dueholm Hansen, and Uri Zwick [41] provided the first lower bound of the form $2^{\Omega(n^\alpha)}$, for some $\alpha > 0$, for both the Random-Edge and the Random-Facet pivot rule in the one-pass variant . This paper was selected the best paper in STOC 2011. They are now in the process of transferring their result (with a slightly worse lower bound) to the original Random Facet algorithm in a forthcoming paper. Using MDPs again and based on [37, 39, 41] Oliver Friedmann constructed an exponential lower bound for the number of steps in Zadeh's rule [40]. Friedmann presented the key points of their construction at IPAM in January 2011. Zadeh, now an entrepreneur running the Perfect 10 website, made time to attend Friedman's presentation. He asked how explicit can one make the construction of the examples. Friedmann noted that although large one can easily write code to generate the problems. At the end of the presentation Zadeh presented the $1000 check to Friedmann (which will remain in custody of David Avis until the referee process for journal publication is finished). See Figure 8. The constructions in both papers amount to the construction of counters of binary sequences. The challenge in designing such counters is making sure that they count correctly all binary sequences of certain size under most sequences of random choices made. Using the combinatorial graphs as building blocks they are able to construct the required sophisticated large MDP problem.

Are these MDP polytopes badly behaved because of their diameter? No, it should be remarked that the diameter of the resulting polytopes is actually linear. In fact, some other MDPs seem to behave quite nicely for other pivot rules. Yinyu Ye [101] showed that the simplex method using Dantzig's pivot rule (where one chooses the entering variable with the largest reduced cost coefficient). is strongly polynomial for the linear programs derived from Markov Decision Processes with Fixed Discount (which is not the setting for the other papers, but is an important case of MDPs). Ye's result inspired others to revisit to the classical pivot rule of Dantzig. Based on Ye's analysis, Kitahara and Mizuno [66, 67], have shown bounds

Figure 8. *Top picture, from left to right: Oliver Friedmann, Thomas Dueholm Hansen, and Uri Zwick (courtesy of Dalya Jacobsen). Bottom picture, from left to right: David Avis, Norman Zadeh, Oliver Friedmann receiving prize check, and IPAM director Russ Caflisch (courtesy of Edward D. Kim)*

for the number of pivots of the simplex method using Dantzig's rules that using the relative sizes of the non-zero coordinates of the vertices: Given a linear program of the form $\max\{c^T x : Ax = b, x \geq 0\}$ where A is a real $d \times n$ matrix, the number of different basic feasible solutions (BFSs) generated by this version of the simplex method is bounded by $n\lceil d\frac{\gamma}{\delta}\log(d\frac{\gamma}{\delta})\rceil$, where δ and γ are the minimum and the maximum values of all the positive elements of primal BFSs and $\lceil a \rceil$ denotes the smallest integer greater than a. Interestingly they also presented a variant of Klee–Minty's LP, for which the number of iterations for this variant is equal to the ratio $\frac{\gamma}{\delta}$.

The results of Friedman et al. came from thinking of pivoting in settings other than linear programming, in their case the theory of games. The fact is that several generalizations of linear programming also have pivoting algorithms whose complexity is similarly unknown. During the IPAM meeting it was evident that other communities would also benefit from the constructions of worst-case pivot behavior. For example, Bernd Gärtner (ETH Zürich), who has worked on pivoting rules for the simplex method, presented a translation of bad-pivot behavior into the realm of machine learning. Putting together some standard techniques and constructions from polytope theory (basically the Goldfarb cube) one can disprove a conjecture of Hastie et al. about the maximum complexity of paths in support vector machines [49].The lesson to be learned is that it is always useful to look at your questions from a different perspective and have knowledge of areas not directly related to yours!

Another topic where pivots are done in a different setting, but still close to the simplex method, is the so called *criss-cross method* [43]. Criss-cross methods are pivoting methods that are allowed to go out of the feasible region, walking not on the graph of it but rather on the graph of the hyperplane arrangement defining it. Thus, instead of focusing on $Ax \leq b$, one considers other sign or direction of the inequalities. Unlike the case for the simplex method, where we do not know polynomial bounds for the diameter of the pivot graph, the diameter of the graph of a hyperplane arrangement is easy to determine, and it is polynomial. Thus the difficulty lies on finding the right pivot rule. Fukuda and Terlaky proved in [44] that in the criss-cross method there is always a sequence of at most n "admissible pivots" (which amounts to certain sign conditions being satisfied) to reach the optimum solution. Fukuda, Terlaky and collaborators have considered other settings such as linear complementarity problems (see [38] and references therein).

Let us conclude by saying that beyond the theoretical analysis of pivot rules there are a number of things we can learn from experiments. For instance, Ziegler [105] reported on studies analyzing some of the well-known NETLIB collection of benchmark problems using the shadow boundary method, a.k.a. two dimensional projection of polytopes. This pivot rule deserves more investigation and plays an interesting role in the smoothed analysis of linear programs [93]. We will discuss more on this in the following section.

2 Results on other Methods

As it is well-known the first polynomial time algorithm for linear programming was the 1979 ellipsoid method of Khachiyan [54]. Later in Karmarkar's paper [62] started the revolution of interior point methods and gave an alternative proof of polynomiality of linear programming. Nevertheless, the question remains: *Is there a strongly polynomial time algorithm which decides the feasibility of the linear system of inequalities $Ax \leq b$?*

As we saw in the previous section, Simplex methods are still contenders to give a positive solution to the question above, but there are in fact many other methods possible beyond the three we mentioned so far. We want to review of some other LP methods, of varied geometric origin, that have been proven to have polynomial-time complexity as well. We discuss a selection of work that took place after 2002. Later in the final part of the article, we will discuss questions about the intrinsic differential geometry of the central curves of interior point methods. Interestingly these geometric inquiries lead to a nice mixture of both discrete and continuous aspects of linear optimization.

2.1 Relaxation & Randomization: Recent efficient algorithms for linear optimization

In their now classical papers ,Agmon [3] and Motzkin and Schoenberg [83] introduced the so called *relaxation method* to determine the feasibility of a system of linear inequalities (it is well-known that optimization and the feasibility problem are polynomially equivalent to one another). Starting from any initial point, a sequence of points is generated. If the current point z_i is feasible we stop, else there must be at least one constraint $a^T x \leq b$ that is violated. The constraint defines a hyperplane H. If w_H is the orthogonal projection of z_i onto the hyperplane H, choose a number λ (normally chosen between 0 and 2), and the new point z_{i+1} is given by $z_{i+1} = z_i + \lambda(w_H - z_i)$.

Many different versions of the relaxation method have been proposed, depending on how the step-length multiplier λ is chosen and which violated hyperplane is used. E.g., the well-known family of perceptron algorithms is part of the method. A bad feature of the standard version of the relaxation method using real-data is that when the system $Ax \leq b$ is infeasible, it cannot terminate, for there will

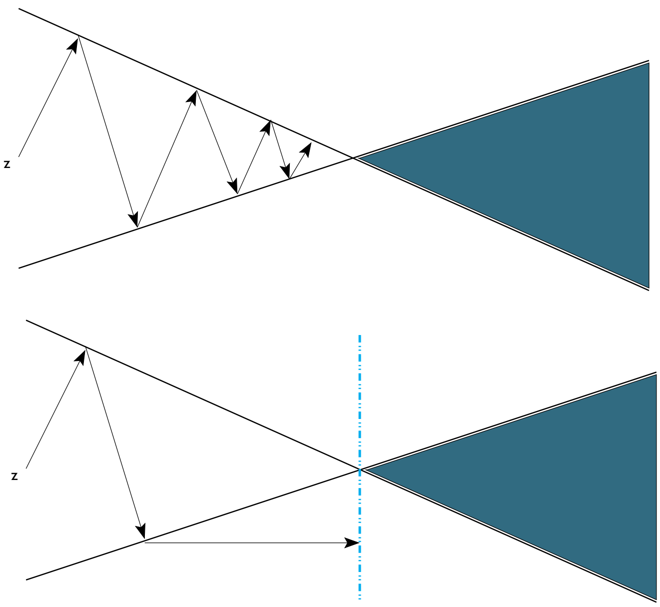

Figure 9. *The standard relaxation method projects only on given constraints (top), Chubanov's variation (bottom) allows for the generation of new constraints that speed up convergence.*

always be a violated inequality. The method was shown early on to have a poor practical convergence to a solution (and in fact, finiteness could only be proved in some cases), thus relaxation methods took second place behind other techniques for years. During the height of the fame of the ellipsoid method, the relaxation method was revisited with interest because the two algorithms share a lot in common in their structure (see [4, 50, 94] and references therein) with the result that one can show that the relaxation method is finite in all cases when using rational data, and thus can handle infeasible systems. In some special cases the method did give a polynomial time algorithm [78], but in general it was shown to be exponential (see [51, 94]).

In 2004, the late Ulrich Betke [16] gave a new efficient version of the relaxation method with rational data. In fact he considers it a mix of ideas used in relaxation and in the ellipsoid method. He works within a spherical homogenized linear feasibility problem (SHFP), i.e., to find an x on the unit sphere S^d satisfying n linear inequalities $a_i^T x \geq 0$. The a_i are assumed to be vectors on the unit sphere, but one can reduce any standard feasibility problem into this form. Let Q be a subset of the normal vectors a_i. The set Q is said to be *nearly positively spanning* if Q is affinely independent and the orthogonal projection O' of the origin O on the affine hull $\operatorname{aff} Q$ is contained in $\operatorname{conv} Q$. The distance of O' to the origin is called the *deficiency* of Q and denoted by $\operatorname{def} Q$. If $k = d + 1$ then Q is a *basis*, but Betke also used the case that $\dim(\operatorname{aff} Q) < d+1$. We can think of Q as the vertices of spherical simplices (some of low dimension), the intuitive idea is to generate a sequence of nearly positive spanning sets Q_k and a point inside them, looking for one that is a feasible solution or finding a certificate of infeasibility, or a way to reduce the dimension of the problem.

For a nearly positively spanning set the deficiency is equal to the distance of the origin to its convex hull. So it may occur that the deficiency is zero, which is an important special case. Betke says a nearly positively spanning set is *positively spanning* if its deficiency is 0. Equivalently Q is positively spanning if and only if Q is affinely independent and $O \in \operatorname{conv} Q$. The positively spanning sets are important for the problem (SHFP) too because, given $Q \subset \{a_1, \ldots, a_n\}$ a positively spanning set, every x in the feasible region in fact satisfies $a_i^T x = 0$ for all $a_i \in Q$. Thus, if the cardinality of Q is $d + 2$, then the instance is infeasible, otherwise the feasible region is contained in the subspace $\{a_i^T x = 0 \mid a_i \in Q\}$ and thus the dimension of the problem is reduced if we run into a positively spanning set of constraints.

Let $Q \subset S^d$ be a set of affinely independent points. Denote by $S(c, R)$ the sphere with center c and radius R. Such a sphere is said to be *touching* for Q if $c \in \operatorname{aff} Q$ and $Q \subset S(c, R)$. Every set of affinely independent points has a unique touching sphere. It is easy to compute a touching sphere for given set Q. The vector set Q is nearly positively spanning if and only if the center of its touching sphere $c \in \operatorname{conv} Q$. In this case $\operatorname{def} Q = \sqrt{1 - R^2}$.

Betke's algorithm generates a sequence Q_k of nearly positively spanning sets. The centers of the touching sphere of Q_k form the desired sequence of points converging to a feasible point (if any). It can be proved that they correspond to the incenters of the spherical simplices given by the constraints a_i. Betke's algorithm is combinatorial as there are only finitely many possible iterates. To force the termination of the algorithm the generation is done in such a way that the deficiency of the Q_k's is strictly decreasing. A set of transformations of the spherical points must be performed to formally make sure the geometric intuition becomes polynomial time.

In late 2010, Sergei Chubanov [23], announced a new polynomial time algorithm to determine the feasibility of a system given in the form $(*)\ Ax = b, Cx \leq f$ with A a full-rank integer $n \times d$ matrix, C an $l \times d$ integer matrix, $b \in \mathbb{Z}^n$, and $f \in \mathbb{Z}^l$. As before let P denote the set of feasible solutions and let $B(z, r)$ denote the open ball centered at z of radius r in $\mathbb{R}^d$. Let a_i denote the i-th row of A and c_k denote the k-th row of C.

At the core of Chubanov's method is a modification of the traditional relaxation style method. The key new idea of Chubanov's algorithm is to construct new inequalities along the way. Unlike [3, 83] who only projected onto the original hyperplanes that describe the polyhedron P, i.e. $c_k x = f_k$, Chubanov [23] projects onto these new auxiliary inequalities. See Figure 9.The new inequalities are linear combinations of the input a_i's and b_i's and nonnegative linear combinations of the c_k's and f_k's, called induced hyperplanes. Thus if $hx = \delta$ is an induced hyperplane, we have

$$h = \sum_{i=1}^{n} \lambda_i a_i + \sum_{k=1}^{l} \alpha_k c_k \quad \text{and} \quad \delta = \sum_{i=1}^{n} \lambda_i b_i + \sum_{k=1}^{l} \alpha_k f_k$$

where the λ_i's and α_k's are real coefficients ≥ 0. The right coefficients of these linear combinations are the result of a recursive call.

The algorithm receives as input the system $Ax = b$, $Cx \leq f$ and a triple (z, r, ϵ). There are three possible outputs for the algorithm: an ϵ-approximate solution x^*, i.e., some x^* such that $Ax^* = b$, $Cx^* \leq f + \epsilon \mathbf{1}$, an induced hyperplane $hx = \delta$ that separates $B(z, r)$ from the solution set, or it returns two induced hyperplanes h_1 and h_2 such that $h_1 = -\gamma h_2$. During the iterations the algorithm behaves as the standard relaxation method (i.e, there are projections into various violated constraints) as long as the radius r is small enough. Otherwise the algorithm recursively calls itself again with smaller and smaller r until the standard procedure can be applied. The induced hyperplanes are not added in any way to the system. As soon as the algorithm moves beyond that level of the recursion, the induced hyperplanes no longer need to be held in memory. Using this modified relaxation algorithm [23] he proved the following intriguing lemma:

Lemma 6. *There exists a strongly polynomial algorithm which either finds a solution of a linear system* $Ax = b$, $0 \leq x \leq 1$, *or correctly decides that the system has no binary* $0, 1$ *solutions.*

To prove that lemma, the second idea of Chubanov is, like in Betke's algorithm, to work on a homogenized system.

$$\begin{aligned} Ax - bt &= \mathbf{0}, \\ Cx - ft &\leq \mathbf{0}, \\ -t &\leq -1. \end{aligned} \tag{1}$$

Note this system (1) is feasible if and only if $(*)$ is feasible. Let (x^*, t^*) be a solution to (1). Then $\frac{x^*}{t^*}$ is a solution of $(*)$. Similarly, if x^* is a solution of $(*)$, then $(x^*, 1)$ is a solution of (1). Chubanov applies his version of the relaxation method to a strengthened parameterized system

$$\begin{aligned} Ax - bt &= \mathbf{0}, \\ Cx - ft &\leq -\epsilon\mathbf{1}, \\ -t &\leq -1 - \epsilon. \end{aligned} \tag{2}$$

The advantage is now that any ϵ-approximate solution will be an exact solution of (1), and thus will give us an exact solution of the original system. The other two outcomes of the Chubanov relaxation method either show a solution of the system or that it cannot have any *integer* solutions. Chubanov's lemma only solves the linear feasibility question or the integer feasibility question in very specific circumstances. An easy reduction can be done to any polyhedron to be able to use Lemma 6. This yields a (non-strongly) polynomial algorithm for linear programming described in [22].

All we have discussed so far are deterministic algorithms, but today it is undeniable that randomization helps in surprising ways to facilitate computation. Linear programming is not immune and we would like to mention some recent situations where randomization plays a big role to get good theoretical guarantees. One first example, still related to the relaxation method we just discussed above, is from the paper by Dunagan and Vempala [34] where they present a randomized version of the perceptron method which they can prove runs in polynomial time.

A second fascinating nice probabilistic algorithm appears in the paper [15]. D. Bertsimas and S. Vempala presented a pretty algorithm for the problem of finding point in a convex set $K \subset \mathbb{R}^d$ specified by a separation oracle (that is a procedure that given a point x, either reports it is in the set K or returns a halfspace that separates the set from x). Clearly for linear programming this is a matter simply checking the constraints of the polyhedron. The key component of the algorithm is sampling the convex set by a random walk. The assumption is that the polyhedron is contained in an axis-aligned cube $P_0 = C$ of width R centered at the origin and, if it is non-empty, then it must contain a cube of width r. Letting $L = \log(R/r)$ the algorithm proceeds as follows: Starting with the center of the containment cube, z_0 and $i = 0$ we check whether $z \in K$. If yes, then we stop, else we let $a^T x \leq b$ be a violated constraint of K. Then, if $H = \{x : a^T x \leq b\}$ we set $P_{i+1} = P_i \cap H$ and increase i by one. Pick N random points $y_1, y_2, \ldots, y_N$ from P_{i+1} and set the new point z_i to be its barycenter (here N is of size polynomial or even linear in the dimension d of the convex set). Repeat until we either find a point inside K or $i > 2dL$ at which point we can report that K is in fact empty.

This algorithm is extraordinarily simple, but there are a number of interesting results that make it possible (the reader may have noted the similarity with the ellipsoid algorithm and its sequence of ellipsoids). First the algorithm works because one can prove that the volume of the enclosing polytopes P_i drop by a constant factor in each iteration. For this the authors proved a beautiful generalization of a classical result of Grünbaum: for a convex set K in $\mathbb{R}^d$, any halfspace that contains the centroid of K, also contains at least $1/e$ of the volume of K. Similarly if after $2dL$ iterations we have not found a point, then the authors proved that with high probability K is empty. The total number of calls for the separation oracle is in fact smaller than those used for the ellipsoid method. Of course one key task is how to actually carry on the sampling of the y_i. It is here that the authors need to use classical results on sampling from convex sets, based on taking random walks on the set K (see [98] and the references therein). The number of steps necessary depends on the geometry of the convex body P_i, thus the authors maintain the "roundness" of the body by applying a suitable affine transformation.

The *smoothed analysis* of algorithms is concerned with the expected running time of an algorithm under slight random perturbations of the input. The smoothed analysis of linear programming by Spielman and Teng [93] (and later with significant improvements by R. Vershynin [99]) have provided new probabilistic insights into why we observe a good practical performance of the simplex algorithm. They demonstrated that on a slight random perturbation of an arbitrary linear program, the simplex method finds the solution after a walk on polytope(s) with expected length polynomial in the number of constraints n, the number of variables d, and the inverse standard deviation of the perturbation $1/\sigma$. Another related development was the 2006 randomized algorithm by Kelner and Spielman [63]. They reduce the input linear program to a special form in which we merely need to certify boundedness. As boundedness does not depend upon the right-hand-side vector, they run the shadow-vertex simplex method with a random right-hand-side vector. Even when the shadow vertex fails it gives a way to modify the perturbation and repeat the process. They proved that the number of iterations is polynomial with high probability. Once more there is nice geometry involved to prove the result. One needs to prove that given a polytope $Ax \leq b$ which is round enough, if one makes a polynomially small perturbation to b, then the number of edges of the projection of the perturbed polytope onto a random two-dimensional subspace is expected to be polynomial. In this setting, one uses the perturbed polytope to decide what to do in the simplex walk of the original data.

The idea of perturbation is a very important development and one can hope that the the same techniques of integral geometry used by Spielman-Teng's smoothed analysis of linear programming and the Spielman-Kelner polynomial algorithm, could perhaps be adapted to show that a Gaussian perturbation of an arbitrary linear program always has expected diameter bounded by the parameters in the program and the inverse of the perturbation variance. This is still to be seen, but at the same time the ideas of smoothed analysis are now also connected to interior point methods in [33] by showing that a slight random relative perturbation of the linear program has small condition number with high probability. In the next subsection we discuss other interesting insights in the geometry of interior point methods.

2.2 Going through the interior: curvature of central paths and interior pivoting

For the rest of the article we will consider the pair of linear programming problems in *primal* and *dual* formulation:

$$\text{Maximize } \mathbf{c}^T\mathbf{x} \text{ subject to } A\mathbf{x} = \mathbf{b} \text{ and } \mathbf{x} \geq 0; \tag{3}$$

$$\text{Minimize } \mathbf{b}^T\mathbf{y} \text{ subject to } A^T\mathbf{y} - \mathbf{s} = \mathbf{c} \text{ and } \mathbf{s} \geq 0. \tag{4}$$

Note that here, unlike the earlier parts of the article, A is an $m \times n$ matrix. The primal-dual interior point methods are among the most computationally successful algorithms for linear optimization. While the simplex methods follow an edge path on the boundary, the

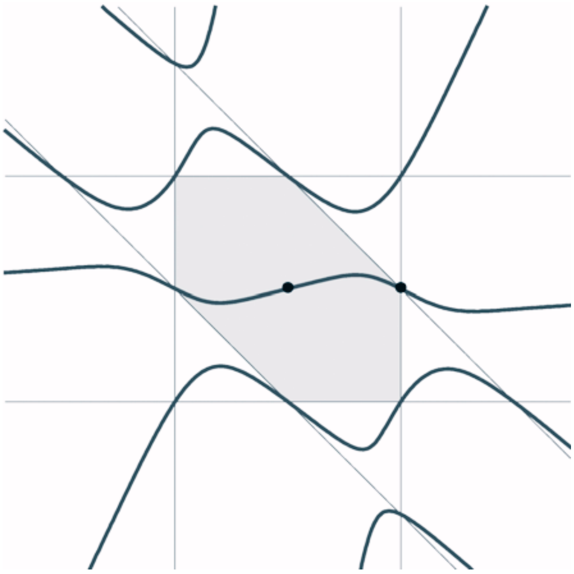

Figure 10. *A view of the entire central curve of a linear program*

interior point methods follow the *central path*. The famous central path is given by the following system of equations

$$A\mathbf{x} = \mathbf{b}, \; A^T\mathbf{y} - \mathbf{s} = \mathbf{c}, \text{ and } x_i s_i = \lambda \text{ for } i = 1, 2, \ldots, n. \quad (5)$$

The system has several properties: For all $\lambda > 0$, the system of polynomial equations has a unique real solution $(\mathbf{x}^*(\lambda), \mathbf{y}^*(\lambda), \mathbf{s}^*(\lambda))$ with the properties $\mathbf{x}^*(\lambda) > 0$ and $\mathbf{s}^*(\lambda) > 0$. The point $\mathbf{x}^*(\lambda)$ is the optimal solution of the *logarithmic barrier function* for (3), which is defined as

$$f_\lambda(\mathbf{x}) \;:=\; \mathbf{c}^T\mathbf{x} \,+\, \lambda \sum_{i=1}^{n} \log x_i.$$

Any limit point $(\mathbf{x}^*(0), \mathbf{y}^*(0), \mathbf{s}^*(0))$ of these solutions for $\lambda \to 0$ is the unique solution of the complementary slackness constraints and thus yields an optimum point. Traditionally the central path is only followed approximately by interior point methods with some kind of Newton steps. Similarly, tradition dictates the central path connects the optimal solution of the linear programs in question with its *analytic center* within one single cell, with $s_i \geq 0$ (a cell of the arrangement is the polyhedron defined by a choice of signs in the constraints).

Nevertheless, from the algebraic-geometric point of view, the traditional central path is just a part of an explicit algebraic curve that extends beyond a single feasibility region (given by sign constraints on variables). Instead of studying the problem with only constraints $s_i \geq 0$, one can ask for all feasible programs arising from any set of sign conditions $s_i \epsilon_i 0$, $\epsilon_i \in \{\leq, \geq\}$, $1 \leq i \leq m$. There are at most $\binom{m-1}{n}$ such feasible sign vectors; (A, b) is said to be in general position if this number is attained. Then the central curve passes through all the vertices of a hyperplane arrangement. See Figure 10. In what follows we talk about the central curve when we wish to emphasize the fact we look at it as an algebraic curve. This kind of thinking goes back to pioneering work by Bayer and Lagarias [11, 12] in the early days of interior point methods.

There are a number of aspects of the central curve one can study. The traditional primal and dual central parts are portions of the projections of the above equations to one single set of variables and one can ask what are the new defining equations. In [27] the authors computed the degree, arithmetic genus and defining prime ideal of the central curve and their primal dual projections. These invariants are expressed in terms of the matroid of the input matrix A, which reinforces the old relationship between (oriented) matroids and linear programming. In practical computations, interior point methods follow a piecewise-linear approximation to the central path.

One way to estimate the number of Newton steps needed to reach the optimal solution is to bound the *total curvature* of the central path. The intuition is that curves with small curvature are easier to approximate with fewer line segments. This idea has been investigated by various authors (see, e.g., [81, 92, 97, 103]), and has yielded interesting results. For example Vavasis and Ye [97] found that the central path contains no more than n^2 turning points. This finding led to an interior-point algorithm whose running time depends only on the constraint matrix A. Thus, in a way, curvature can be regarded as the continuous analogue of the role played by the diameter in the simplex method.

In [28] Dedieu, Malajovich, and Shub investigated the differential geometric properties of the central curve of interior point methods. Their main theorem is as follows: Let (A, b, c) be as above with (A, b) in general position. Then the *average* total curvature of the primal, the dual, and the primal-dual central paths of the strictly feasible polytopes defined by (A, b) is at most $2\pi(n-1)$ (primal), at most $2\pi n$ (dual), and at most $2\pi n$ (primal-dual), respectively. In particular, it is independent of the number m of constraints. The bounds they obtained come from first reducing the curvature estimates to an integral geometry problem (expected number of intersections of the Gauss curve with a random plane), and then to the estimation of the number of roots of a particular polynomial system. More recently [27] obtained bounds for the total curvature in terms of the degree of the Gauss maps of the curve. For the interesting case of two dimensions the total curvature of a plane curve can be bounded in terms of the number of real inflection points, and they derive a new bound from a classical formula due to Felix Klein which gives a slight improvement to the bound in [28].

Of course, for practical applications the more relevant quantity is not the average total curvature but rather the curvature in a single feasible region! This has been investigated by A. Deza, T. Terlaky and Y. Zinchenko in a series of papers. Dedieu et al. conjectured that the curvature (in a single cell) could only grow linearly in the dimension. Deza, Terlaky and Zinchenko [30] constructed central paths that are forced to visit small neighborhoods near of all vertices of a cube, "a la Klee-Minty". Their construction shows the Dedieu et al. conjecture is false. In [29] they proved that even for $d = 2$ the total curvature grows linearly in the number of facet constraints. They conjectured the following curvature analogue of diameter:

Conjecture 7 (continuous Hirsch conjecture)**.** *The curvature of a polytope, defined as the largest possible total curvature of the associated central path with respect to the various cost vectors, is no more than $2\pi m$, where m is the number of facets of the polytope.*

The name of the conjecture suggests the similarity with the discrete simplex method. Deza, Terlaky and Zinchenko (see [31]) investigated other analogies between the continuous and combinatorial methods to linear optimization They proved a continuous analogue of the "d-step theorem" of Klee and Walkup, saying that to prove the continuous Hirsch conjecture in general you only need to establish it in the case when the number of constraints is twice the dimension. It is worth noting that their construction in [30] does not contradict the continuous Hirsch conjecture, since they need to add exponentially many (redundant) constraints to force the central path do what they want, and thus the number of constraints is larger than the number of facets. Although the average value for the curvature for bounded cells is known to be linear, we do not have a polynomial bound for the total curvature in a single cell. During the IPAM conference Yuriy Zinchenko presented a conjecture, by him, Deza and Terlaky, made in 2006: The central path curvature of the feasible region cannot exceed $\pi d(n-d)$. It is somewhat similar to the more recent Hähnle conjecture. By the way, since we have good linear bounds for the *average* total curvature of the central path, we should also think about the *average diameter* among bounded cells of a hyperplane arrangement! Thanks to Santos' counterexample the diameter of polytopes is not always equal to the predicted difference between number of facets and the dimension, but it might still be true on average, a problem suggested by Deza et al. (see [29]).

Another development in the geometry of interior point methods for linear programming came from the notions of hyperbolic polynomials and hyperbolic cones. A homogeneous polynomial $p(\cdot)$ is said to be *hyperbolic* if there exists a direction e with $p(e) \neq 0$ and that for each vector x, the univariate polynomial $t \mapsto p(x+te)$ has only real roots. The roots of the polynomial $p(\lambda e - x)$ are called the *eigenvalues* of x. An interesting geometric object is the the *hyperbolicity cone* of a (hyperbolic) polynomial; this is the set of all points x with positive eigenvalues. The hyperbolicity cone is a convex cone. A hyperbolic program aims to minimize a linear functional over an affine subspace of a real space intersected with the hyperbolicity cone of a polynomial p. This is a class of convex optimization problems that contains linear programming. Indeed, the traditional set up of standard linear programs $\max c^T x, Ax = b, x \geq 0$ is using the hyperbolicity cone of the polynomial $p = x_1 x_2 \ldots x_n$ with respect to the all-ones vector. All vectors in the positive orthant are part of the hyperbolicity cone. The cone of positive definite matrices is another key example of hyperbolic cone, with $p = det(x)$. The hyperbolic program in this case is a semidefinite program.

To solve a linear program the idea in [85, 86] is use hyperbolic programs to create a sequence of relaxations to the initial linear program. Taking $p(x) = x_1 x_2 \ldots x_n$ one must consider the high-order univariate derivatives of the polynomial $p(x+te)$ then evaluate them at $t = 0$. If p was hyperbolic in direction e, the resulting derivatives are also hyperbolic in the direction e. It turns out the hyperbolicity cone defined by the i-th derivative polynomial contains the original cone. Thus by taking higher-order derivatives of the polynomial p, a nested family of cones can be obtained and the cones become tamer as additional derivatives are taken, the largest cone being just an open halfspace. As one moves closer toward the original cone the cones start resembling the non-negative orthant. From these cones one tries to recover the LP solution. For instance, depending on the choice of the direction e, it can happen that the i-th hyperbolic relaxation has no optimal solution (problem becomes unbounded). So we can define the i-th *central swath* as $Sw(i)$ the set of directions of feasible directions e which have an optimal solution in the i-th hyperbolic relaxation. The traditional central path turns out to be the $(n-1)$-th central swath, but all others swaths are not necessarily one-dimensional curves. They can be used to create a sequences of points going to the optimum solution, where the steps of iteration are guided by the swath points, until one converges to the optimum of the linear program. In a way, these generalize interior point methods as the step at each iteration is not restricted to follow only the central path. See [86].

To conclude this survey we would like to mention attempts to use what one would call "combinatorial interior point methods". Unlike the simplex method (that makes steps along the edges of polyhedron) or the interior point methods (that follow a non-linear path) the method takes linear steps in directions that belong to the faces of feasible region or across its interior. For instance, [21] presented such a method with an experimental implementation with interesting results. More recently Barasz and Vempala [8] presented another example. Their main idea is to walk along a piece-wise linear path. At the beginning of each step one is at a feasible vertex and must shoot a ray through the interior of the polytope averaging between the increasing edges that we have at hand. The average ray can be uniform, or randomized. One moves along this ray until hitting a facet of the polytope, one then finds a way of getting back to a vertex. They have proved their algorithm is strongly polynomial when applied to the notoriously difficult deformed products of [5].

It will be interesting to see more results on all these topics, the future is full of new possibilities and interesting problems!

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Discussion Column

Günter M. Ziegler

Comments on *New Insights into the Complexity and Geometry of Linear Optimization*

When in May 2010 Francisco Santos announced his counter-example to the Hirsch conjecture, this was immediately picked up on Gil Kalai's blog, where a desperate Stanford graduate student posted "That's my whole PhD work going to trash!".

Although this may sadly have been true if the thesis project was very specific, say one of many clever attempts to prove the conjecture, Kalai and Santos immediately responded that the counter-example certainly shouldn't discourage the student or anyone else to continue work on this topic, or to join the community. Indeed, Santos' breakthrough is not one that kills a field by solving its key problem, but it puts the spotlight onto a central and important research area, and hopefully gets many people (included, but not restricted to, clever Stanford graduate students) to work on exactly that. Even after Santos' breakthrough, the lower bound for the maximal diameter of the graphs of polytopes is linear in the number of facets/variables, the upper bound is exponential (if, say, the dimension is fixed). In particular, we still have no idea about worst-case complexity of an optimal simplex algorithm.

There had been very little visible progress in the analysis of the Hirsch conjecture and the pertinent combinatorics of polytopes and their graphs since Kalai's subexponential bounds more than twenty years ago and Spielman & Teng's work on smoothed analysis more than ten years ago. Thus we have every right for enthusiasm in a situation where several unexpected substantial breakthroughs from quite different directions surface nearly simultaneously – as outlined in Jesús De Loera's survey. The Hirsch conjecture still poses the greatest challenge in the area, namely to determine the asymptotics of the function that measures the maximal graph diameter for a d-dimensional polytope with n facets. Besides that, I am most excited by the work by Oliver Friedmann and coauthors about exponential lower bounds for various randomized pivot rules, which may have more of a danger of "killing the field" by solving all the key problems – but on the other hand opens a powerful new avenue in the connection to Markov decision processes.

How far have we gotten in our understanding of convex polytopes and the simplex algorithm and the complexity of linear programming? My claim would be that "you ain't seen nothing yet". We have no real understanding of the geometry, the combinatorics, topological aspects and the spectral properties of high-dimensional polytopes. All these properties are relevant – for example, Santos' work has a distinctive topological core, which remains to be explored further. Random processes ask for spectral analysis, as expansion properties govern the behaviour of random walks on the graph. Progress will come from bringing these aspects together.

One aspect I keep wondering about is the geometry of high-dimensional polytopes/problems. In what sense are high-dimensional polytopes "round"? Indeed, the upper bounds on subdeterminants imposed by Bonifas et al. translate into lower bounds on the volumes in normal cones, which may be interpreted as an (integral) version of roundness (namely, lower bounds on curvature), and "round" polytopes may behave differently from others. For example, according to work by Figiel et al. (1977) and very recent updates by Barvinok (arXiv, August 2011), sufficiently round polytopes have many facets and/or many vertices. Also, celebrated work by Bárány & Pór (2001) is based on the intuition that high-dimensional random 0/1 polytopes are quite "round" (approximating a convex body with lower curvature bounds) and because of this have more than exponentially many facets. On the other hand, Milman and Kalai have suggested that we should picture high-dimensional polytopes like amoebae with tentacles, since so little of their volume is concentrated near to the vertices. My own study of 2-dimensional shadows of "real life" linear programs shows both round and pointed characteristics. How do typical polytopes look like? How do real-life linear programs look like? How do extremal polytopes look like? Don't take the 3-cube (or the 3D Klee–Minty cube) as a realistic image!

Another remarkable aspect is the great variety of algorithmic ideas beyond the "good old simplex algorithm," both old and new, now come into play, possibly should be connected. These are as diverse as the criss-cross paradigm for the simplex algorithm, various interior point strategies, but also ellipsoid algorithms, relaxation ideas, etc. But are they really separate? How are they connected, how do they compare? Can you, for example, obtain Chubanov's polynomial time algorithm using the ellipsoid method? This was suggested at IPAM by Fritz Eisenbrand. Not only new algorithmic ideas, but also old methods need to be reevaluated and connected.

De Loera's paper is a theory survey. Does the work he reports about have practical significance? I would say that the answer is clearly *no* for now – but also clearly *yes*, it will. Santos says that his work "breaks a psychological barrier, but for applications it is absolutely irrelevant." This remains to be seen: Let me remind you that "Gomory cuts" were invented in the early sixties as a purely theoretical tool for finiteness proofs in integer programming; practical tests proved them to be utterly useless. In the nineties they were re-evaluated by Balas, Ceria and Cornuéjols and to everyone's greatest surprise found to be excellent. They were then immediately integrated into commercial and public domain libraries codes for integer and mixed-integer programming. Perhaps similarly, the fact that TSP-polytopes have very small diameter was declared to be utterly useless when it was discovered by Padberg & Rao (1974). Is it really? Thus, the impact of the new ideas sketched in this survey remains to be seen, and it will be seen, sooner or later. If we work hard, sooner.

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Announcements

MIP 2012

We are pleased to announce that the 2012 workshop in Mixed Integer Programming (MIP 2012) will be held July 16–19, 2012 at the University of California, Davis. The 2012 Mixed Integer Programming workshop will be the ninth in a series of annual workshops held in North America designed to bring the integer programming community together to discuss very recent developments in the field. The workshop series consists of a single track of invited talks and also features a poster session as an additional opportunity to share and discuss recent research. Registration details and a call for participation in the poster session will be announced later.

Confirmed speakers

Gennadiy Averkov (Otto-von-Guericke-Universität Magdeburg), Sam Burer (The University of Iowa), Philipp Christophel (SAS), Jesús A. De Loera (University of California (Davis), Alberto Del Pia (ETH Zurich), Friedrich Eisenbrand (EPFL), Ricardo Fukasawa (University of Waterloo), Vineet Goyal (Columbia University), Yongpei Guan (University of Florida), Volker Kaibel (Otto-von-Guericke-Universität Magdeburg), Kiavash Kianfar (Texas A&M University), Mustafa Kılınç (University of Pittsburgh), Fatma Kılınç-Karzan (Carnegie Mellon University), David Morton (The University of Texas at Austin), Ted Ralphs (Lehigh University), Edward Rothberg (Gurobi Optimization), Siqian Shen (University of Michigan), Renata Sotirov (Tilburg University), Dan Steffy (ZIB and Oakland University), Alejandro Toriello (University of Southern California), Christian Wagner (ETH Zurich)

Claudia D'Ambrosio, CNRS – École Polytechnique
Matthias Köppe, UC Davis
Jim Luedtke, University of Wisconsin-Madison
François Margot, Carnegie Mellon University
Juan Pablo Vielma, University of Pittsburgh

(MIP 2012 Organizing Committee
mip2012@math.ucdavis.edu)

Further information: http://www.math.ucdavis.edu/mip2012/

Third Conference on Optimization Methods and Software

Crete, May 13–17, 2012. This international conference aims on bringing together leading experts in the fields of optimization and computational methods to discuss recent advancements and trending topics.

Dates and Deadlines:

Abstract submissions: January 15, 2012
Notification of acceptance: February 1, 2012
Early registration: March 31, 2012

Registration Fee:

Early registration: EUR 350 (EUR 150 for students and accompanying persons); late registration: EUR 400 (EUR 200 for students and accompanying persons).

Conference Organizers:

Oleg Burdakov (Sweden), Conference Chair; Panos Pardalos (USA), Conference Chair; Luis Nunes Vicente (Portugal), Program Committee Chair, Alper Yildirim (Turkey), Organizing Committee Chair; Masao Fukushima (Japan); Andreas Griewank (Germany); Michael Hintermüller (Germany); Michal Kocvara (GB); Yurii Nesterov (Belgium); Panos Pardalos (USA); Liqun Qi (China); Andrzej Ruszczynski (USA); Ekkehard Sachs (Germany); Marco Sciandrone (Italy); Philippe Toint (Belgium); Stefan Ulbrich (Germany); Luis Nunes Vicente (Portugal); Yinyu Ye (USA); Alper Yildirim (Turkey); and Ya-xiang Yuan (China).

Further information: http://www.ise.ufl.edu/cao/OMS12/OMS_2012/

ISMP 2012 in Berlin

The organizers of the *21st International Symposium on Mathematical Programming (ISMP 2012)* have the great pleasure of inviting you to Berlin, Germany, August 19–24, 2012. ISMP is the world congress of mathematical optimization and is held every three years on behalf of the Mathematical Optimization Society (MOS).

Call for Sessions and Presentations

Presentations on all theoretical, computational and practical aspects of mathematical programming in one of the clusters listed below are welcome. Invited and contributed presentations will be organized in parallel sessions, each session consisting of three talks. Interested session organizers are invited to contact the cluster chairs for their particular topic.

The deadline for submitting invited sessions is March 1, 2012. The deadline for submitting titles and abstracts of presentations is April 15, 2012. There is a one talk per speaker policy at ISMP, i.e., no participant will be able to register more than one presentation. Abstract submission will be done online, via the conference web page at www.ismp2012.org.

Clusters and Cluster Chairs

- Approximation and Online Algorithms (Leen Stougie, David P. Williamson)
- Combinatorial Optimization (Jochen Könemann, Jens Vygen)
- Complementarity and Variational Inequalities (Michael C. Ferris, Michael Ulbrich)
- Conic Programming (Raphael Hauser, Toh Kim Chuan)
- Constraint Programming (Michela Milano, Willem-Jan van Hoeve)
- Derivative-free and Simulation-based Optimization (Luis Nunes Vicente, Stefan Wild)
- Finance and Economics (Thomas F. Coleman, Karl Schmedders)
- Game Theory (Asu Ozdaglar, Guido Schäfer)
- Global Optimization (Christodoulos A. Floudas, Nikolaos V. Sahinidis)
- Implementations and Software (Tobias Achterberg, Andreas Wächter)
- Integer and Mixed-Integer Programming (Andrea Lodi, Robert Weismantel)
- Life Sciences and Healthcare (Gunnar W. Klau, Ariela Sofer)
- Logistics, Traffic, and Transportation (Marco E. Lübbecke, Georgia Perakis)
- Mixed-Integer Nonlinear Progamming (Sven Leyffer, François Margot)
- Multi-Objective Optimization (Jörg Fliege, Johannes Jahn)
- Nonlinear Programming (Philip E. Gill, Stephen J. Wright)
- Nonsmooth Optimization (Amir Beck, Jérôme Bolte)
- Optimization in Energy Systems (Alexander Martin, Claudia Sagastizábal)

Venue of the ISMP 2012 opening ceremony: Konzerthaus am Gendarmenmarkt, Berlin (Photo: Christoph Eyrich)

- PDE-constrained Optimization and Multi-Level/Multi-Grid Methods (Matthias Heinkenschloss, Michael Hintermüller)
- Robust Optimization (Aharon Ben-Tal, Dimitris Bertsimas)
- Sparse Optimization and Compressed Sensing (Michael Saunders, Yin Zhang)
- Stochastic Optimization (Shabbir Ahmed, David Morton)
- Telecommunications and Networks (Andreas Bley, Mauricio G. C. Resende)
- Variational Analysis (René Henrion, Boris Mordukhovich)

Plenary and Semi-Plenary Speakers

Nikhil Bansal (TU Eindhoven), Richard G. Baraniuk (Rice University), Dimitris Bertsimas (Massachusetts Institute of Technology), Xiaojun Chen (Hong Kong Polytechnic University), Michael P. Friedlander (University of British Columbia), Jorge Nocedal (Northwestern University), Teemu Pennanen (University of Jyväskylä), Amin Saberi (Stanford University), Claudia Sagastizábal (CEPEL (Rio de Janeiro), Katya Scheinberg (Lehigh University), Christof Schütte (Freie Universität Berlin), Rekha R. Thomas (University of Washington), Robin Thomas (Georgia Institute of Technology), Rakesh V. Vohra (Northwestern University), and Robert Weismantel (ETH Zürich).

Important Dates

March 1, 2012:	Deadline for submitting invited sessions
April 15, 2012:	Abstract submission deadline
June 15, 2012:	Early registration deadline
August 19, 2012:	Opening ceremony at *Konzerthaus am Gendarmenmarkt*

MOPTA 2012

July 30 – August 1, 2012 Lehigh University, Rauch Business Center, Bethlehem, PA 18015

MOPTA aims at bringing together a diverse group of people from both discrete and continuous optimization, working on both theoretical and applied aspects. There will be a small number of invited talks from distinguished speakers and contributed talks, spread over three days. Our target is to present a diverse set of exciting new developments from different optimization areas while at the same time providing a setting which will allow increased interaction among the participants. We aim to bring together researchers from both the theoretical and applied communities who do not usually have the chance to interact in the framework of a medium-scale event.

Confirmed plenary speakers

Kurt M. Anstreicher (U. of Iowa), Edgar Blanco (MIT-CTL), Andrew V. Goldberg (Microsoft Research), Mark S. Roberts (U. of Pittsburgh), Michael Trick (Carnegie Mellon), Santosh S. Vempala (Georgia Tech), Henry Wolkowicz (U. of Waterloo), and Stephen J. Wright (U. of Wisconsin).

Organizing Committee

Luis F. Zuluaga (Chair), Tamás Terlaky, Katya Scheinberg, Ted Ralphs, Robert Storer, Aurélie Thiele, Larry Snyder, Eugene Perevalov, and Frank E. Curtis.

Further information

http://coral.ie.lehigh.edu/~mopta/

We look forward to seeing you at MOPTA 2012!

IMPRINT
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